

On the Asymptotic Expansion of Complete Ricci-flat Kähler metrics on quasi-projective manifolds

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Abstract

In this work, we describe the asymptotic behavior of complete metrics with prescribed Ricci curvature on open Kähler manifolds that can be compactified by the addition of a smooth and ample divisor. First, we construct a explicit sequence of Kähler metrics with special approximating properties. Using those metrics as starting point, we are able to work out the asymptotic behavior of the solutions given in [TY1], in particular obtaining their full asymptotic expansion.

1 Introduction

In 1978, Yau [Y] proved the Calabi Conjecture by showing the existence and uniqueness of Kähler metrics with prescribed Ricci curvature on compact complex manifolds.

Following this work, Tian and Yau [TY1] settled the non-compact version of Calabi's Conjecture on quasi-projective manifolds that can be compactified by adding a smooth, ample divisor. In a subsequent work ([TY2]), their result was extended to the case where the divisor has multiplicity greater than one and orbifold-type singularities. This generalization was independently done by Bando [B] and Kobayashi [K].

Once the existence problem is solved, an interesting question that arises concerns the behavior of these complete metrics near the divisor. This question is also addressed to by Tian and Yau ([TY1]).

In a subsequent work, Tian and Yau ([TY2]) obtain existence results, as well as the asymptotics of their solution, for the case in which the divisor has multiplicity strictly greater than one. These techniques, however, make an essential use of the higher multiplicity of the divisor and thus they do not work when this multiplicity equals one which can be thought of as the generic case. In particular the asymptotics properties of the solutions found in ([TY1]) remained ignored.

The aim of this paper is to provide an answer to this question, therefore refining the main result in [TY1]. More precisely, we shall first construct a sequence of complete Kähler metrics with special approximating properties on a quasi-projective manifold (in our case, the complement of a smooth, ample divisor on a compact complex manifold). Then by using these approximating metrics, we are going to study the solution of a complex Monge-Ampère equation on the open manifold. A careful analysis of the complex Monge-Ampère operator will allow us to describe the asymptotic properties of the solution. As a matter of fact, the reader will note that our results apply equally well to divisors having orbifold-type singularities.

In a subsequent work, we expect to remove the smoothness assumption on the divisor so as to include divisors having normal crossings.

To state the main results of this paper, let us consider a compact, complex manifold \overline{M} of complex dimension n . Let D be an *admissible* divisor in \overline{M} , ie, a divisor satisfying the following conditions:

- $\text{Sing } \overline{M} \subset D$.
- D is smooth in $\overline{M} \setminus \text{Sing } \overline{M}$.
- For every $x \in \text{Sing } \overline{M}$, the corresponding local uniformization $\Pi_x : \tilde{\mathcal{U}}_x \rightarrow \mathcal{U}_x$, with $\tilde{\mathcal{U}}_x \in \mathbb{C}^n$, is such that $\Pi_x^{-1}(D)$ is smooth in $\tilde{\mathcal{U}}_x$.

Let Ω be a smooth, closed $(1, 1)$ -form in the cohomology class $c_1(K_{\overline{M}}^{-1} \otimes L_D^{-1})$, where $K_{\overline{M}}$ stands for the Canonical Line Bundle of \overline{M} , and L_D for the line bundle associated to D . Let S be a defining section of D on L_D and let M be the open manifold $M = \overline{M} \setminus D$. Consider a hermitian metric $||\cdot||$ on L_D .

Fefferman, in his paper [F], developed inductively an n -th order approximation to a complete Kähler-Einstein metric on strictly pseudoconvex domains on \mathbb{C}^n with smooth boundary, and he suggested that higher order approximations could be obtained by considering log terms in the formal expansion of the solution to a certain complex Monge-Ampère equation. This idea was used by Lee and Melrose in [LM], where they constructed the full asymptotic expansion of the solution to the Monge-Ampère equation introduced by Fefferman.

Motivated by this work, we construct inductively a sequence of rescalings $||\cdot||_{\phi_m} := e^{\phi_m/2}||\cdot||$ of a fixed hermitian metric $||\cdot||$ on L_D , which will be the main ingredient of the proof of the following result.

Theorem 1.1 *Let M , Ω and D be as above. Then, for every $\varepsilon > 0$, there exists an explicitly given complete Kähler metric g_ε such that*

$$\text{Ric}(g_\varepsilon) - \Omega = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} f_\varepsilon \quad \text{on } M, \quad (1)$$

where f_ε is a smooth function on M such that all its covariant derivatives decay to the order of $O(||S||^\varepsilon)$. Furthermore, for any $k \geq 0$, the norm of the k -th covariant derivative of the Riemann curvature tensor $R(g_\varepsilon)$ of the metric g_ε decays to the order of $O((-n \log ||S||^2)^{-\frac{k+2}{2n}})$.

Remark: In the above statement, it should be emphasized that the metric in question is constructed inductively in an explicit manner. In other words, this result provides complete metrics that are “approximate solutions” to the Calabi problem, but that have the advantage of being explicitly described.

So far, there has been a large amount of work concerned with deriving asymptotic expansions for Kähler-Einstein metrics in different contexts: after Cheng and Yau [CY1] proved existence and uniqueness of Kähler-Einstein metrics on strictly pseudoconvex domains in \mathbb{C}^n with smooth boundary (in addition to results on the regularity of the solution), Lee and Melrose [LM] derived an asymptotic expansion for the Cheng-Yau solution, which completely determines the form of the singularity and improves the regularity result of [CY1]. On the setting of quasi-projective manifolds, Cheng and Yau [CY2] and Tian and Yau [TY3] showed the existence of Kähler-Einstein metrics under certain conditions on the divisor, and Wu developed the asymptotic expansion to the Cheng-Yau metric on a quasi-projective manifold (also assuming some conditions on the divisor), as the parallel part to the work of Lee and Melrose [LM].

However the asymptotic description of complete Kähler metrics with prescribed Ricci curvature, which are not of Kähler-Einstein type, was still lacking, for example in the context of quasi-projective manifolds considered in ([TY1]). This description will be provided by our results below.

In [TY1], the result of existence of a complete Kähler metric (in a given Kähler class) with prescribed Ricci curvature is achieved by solving the following complex Monge-Ampère equation

$$\begin{cases} \left(\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u \right)^n = e^f \omega^n, \\ \omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u > 0, \end{cases} \quad u \in C^\infty(M, \mathbb{R}), \quad (2)$$

where f is a given smooth function satisfying the integrability condition

$$\int_M (e^f - 1) \omega^n = 0. \quad (3)$$

Our main result describes the asymptotic behavior of the solution to (2), by showing that the approximate metrics given in Theorem 1.1 are asymptotically as close to the actual solution as possible.

We should point out here that the underlying analysis of the case we are considering, as well as the methods used in our work, differs fundamentally from Wu’s work ([W]). Namely, it is not trivial to show, for example, that the solution $u(x)$ to (2) vanishes uniformly when x approaches infinity. In the Kähler-Einstein context considered by Wu, it is not hard to see it directly from the defining equation (as

observed by Cheng and Yau in [CY1]). Also, the log-filtration of the Cheng-Yau Hölder ring considered by Wu is not preserved in our case.

Theorem 1.2 *For each $\varepsilon > 0$, let g_ε and f_ε be given by Theorem 1.1.*

Consider the problem

$$\begin{cases} \left(\omega_{g_\varepsilon} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u_\varepsilon \right)^n = e^{f_\varepsilon} \omega_{g_\varepsilon}^n, \\ \omega_{g_\varepsilon} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u_\varepsilon > 0, \end{cases} \quad u_\varepsilon \in C^\infty(M, \mathbb{R}). \quad (4)$$

Then, there exists a solution $u_\varepsilon(x)$ which decays to the order of at least $O(\|S\|^\varepsilon(x))$ for x sufficiently close to D .

Moreover, the norm of the k -th covariant derivative $|\nabla^k u_\varepsilon|_{g_\varepsilon}$ of u_ε decays as $O(\|S\|^\varepsilon(x) \rho_{g_\varepsilon}^{-\frac{k+2}{n+1}})$, where $\rho_{g_\varepsilon}(x)$ denotes the distance, with respect to the metric g_ε , from a fixed point $x_0 \in M$ to x .

This theorem has an important, straightforward corollary.

Corollary 1.1 *Let \overline{M} be a compact Kähler manifold of complex dimension n , and let D be a smooth anti-canonical divisor. Then for any $\varepsilon > 0$, there exists a complete Ricci-flat Kähler metric on $M = \overline{M} \setminus D$ that can be described as*

$$\hat{\omega} = \omega_{g_\varepsilon} + \omega_{u_\varepsilon},$$

where g_ε is the Kähler metric constructed in Theorem 1.1, and ω_{u_ε} is a Kähler form that decays at least to the order of $O(\|S\|^\varepsilon)$ when x approaches the divisor. Therefore, ω_{g_ε} provides the asymptotics of the Ricci-flat metric $\hat{\omega}$.

The structure of the paper is as follows. In Section 2, we construct inductively a sequence of hermitian metrics $\{\|\cdot\|_m\}_{m \in \mathbb{N}}$ on L_D such that the closed $(1,1)$ -form

$$\omega_m = \frac{\sqrt{-1}}{2\pi} \frac{n^{1+1/n}}{n+1} \partial \bar{\partial} (-\log \|S\|_m^2)^{\frac{n+1}{n}} \quad (5)$$

is positive definite on a tubular neighborhood V_m of D in \overline{M} .

The Kähler form ω_m defines a Kähler metric g_m on V_m such that $\text{Ric}(g_m) - \Omega = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} f_m$, for a smooth function f_m on M that decays to the order of $\|S\|^m$. An important technical result for this construction is Lemma 2.2, whose proof is the object of Section 3.

In Section 4, we use the constructions of Section 2 to complete the proof of Theorem 1.1. First, we shall obtain the necessary estimates on the decay of the Riemann curvature tensor of the metrics g_m . Then we shall proceed to the construction of approximating metrics that are defined on the whole manifold (and not only on a neighborhood of the divisor at infinity).

Finally, Section 5 is devoted to the asymptotic study of the Monge-Ampère equation (2). By using the maximum principle for the complex Monge-Ampère operator, along with the construction of a suitable barrier, we shall complete the proof of Theorem 1.2.

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2 Approximating Kähler metrics

Let \overline{M} be a compact Kähler manifold of complex dimension n , and let D be an admissible divisor in \overline{M} .

The divisor D induces a line bundle L_D on \overline{M} . We will assume that the restriction of L_D to D is ample, so that there exists an orbifold hermitian metric $\|\cdot\|$ on L_D such that its curvature form $\tilde{\omega}$ is positive definite along D .

Consider a closed $(1,1)$ -form Ω in the Chern class $c_1(-K_{\overline{M}} - L_D)$. The goal of this section is to construct a complete kähler metric g such that

$$\text{Ric}(g) - \Omega = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} f \text{ on } M, \quad (6)$$

for a smooth function f with sufficiently fast decay, where $M = \overline{M} \setminus D$ and $\text{Ric}(g)$ stands for the Ricci form of the metric g .

Fix an orbifold hermitian metric $||\cdot||$ on L_D such that its curvature form $\tilde{\omega}$ is positive definite along D . We shall need to rescale the metric by a suitable factor which will be determined in the following discussion. Let us begin by observing that the restriction $\Omega|_D$ of Ω to D belongs to $c_1(D)$ since, by assumption, $\Omega \in c_1(-K_{\overline{M}} - L_D)$. Hence, there exists a function φ such that $\tilde{\omega}|_D + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi$ defines a metric g_D verifying $\text{Ric}(g_D) = \Omega|_D$. So, by rescaling $||\cdot||$ by an appropriate factor, we may assume that $\tilde{\omega}$, when restricted to the infinity D , defines a metric g_D such that $\text{Ric}(g_D) = \Omega|_D$.

Next denote by S the defining section of D , and write $||\cdot||_\phi = e^{-\phi/2} ||\cdot||$ for the rescaling of $||\cdot||$, where ϕ is any smooth function on \overline{M} .

We define

$$\omega_\phi = \frac{\sqrt{-1}}{2\pi} \frac{n^{1+1/n}}{n+1} \partial \bar{\partial} (-\log ||S||_\phi^2)^{\frac{n+1}{n}}. \quad (7)$$

Then it follows that

$$\omega_\phi = (-n \log ||S||_\phi^2)^{1/n} \tilde{\omega}_\phi + \frac{1}{(-n \log ||S||_\phi^2)^{(n-1)/n}} \frac{\sqrt{-1}}{2\pi} \partial \log ||S||_\phi^2 \wedge \bar{\partial} \log ||S||_\phi^2, \quad (8)$$

where $\tilde{\omega}_\phi$ is the curvature form of the metric $||\cdot||_\phi$. From this expression, we can see that, as long as $\tilde{\omega}_\phi$ is positive definite along D , ω_ϕ is positive definite near D .

We state here the main result of this section.

Proposition 2.1 *Let \overline{M} be a compact Kähler manifold of complex dimension n , and let D be an admissible divisor in \overline{M} . Consider also a form $\Omega \in c_1(-K_{\overline{M}} - L_D)$, where L_D is the line bundle induced by D .*

Then there exist sequences of neighborhoods $\{V_m\}_{m \in \mathbb{N}}$ of D along with complete Kähler metrics ω_m on $(V_m \setminus D, \partial(V_m \setminus D))$ (as defined in (7)) such that

$$\text{Ric}(\omega_m) - \Omega = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} f_m \text{ on } V_m \setminus D \quad (9)$$

where f_m are smooth functions on $M = \overline{M} \setminus D$. Furthermore each f_m decays to the order of $O(||S||^m)$. In addition the curvature tensors $R(g_m)$ of the metrics g_m decay at least to the order of $(-n \log ||S||^2)^{\frac{-1}{n}}$ near the divisor.

The remainder of this section will be devoted to the proof of Proposition 2.1.

If $\tilde{\omega} = -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log ||S||^2$ is the curvature form of $||\cdot||$, then for any Kähler metric g' on \overline{M} , $\text{Ric}(g') - \tilde{\omega} \in c_1(-K_{\overline{M}} - L_D)$. Hence, up to constant, there is a unique function Ψ such that

$$\Omega = \text{Ric}(g') - \tilde{\omega} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \Psi. \quad (10)$$

Definition 2.1 *For x in the set where ω_ϕ is positive definite (x near D), write*

$$f_\phi(x) = -\log ||S||^2 - \log\left(\frac{\omega_\phi^n}{\omega'^n}\right) - \Psi,$$

where ω' is the Kähler form of g' .

Lemma 2.1 *The function $f_0(x)$ converges uniformly to a constant if and only if $\text{Ric}(g_D) = \Omega|_D$.*

Proof: Choose a coordinate system (z_1, \dots, z_n) around a point x near D such that the local defining section S of D is given by $\{z_n = 0\}$. In these coordinates, write $\tilde{\omega} = \tilde{\omega}_0$ as $(h_{ij})_{1 \leq i, j \leq n}$, g' as $(g'_{ij})_{1 \leq i, j \leq n}$, and $\|\cdot\|$ as a positive function a .

By definition we have

$$f_0(x) = -\log \left(\frac{\|S\|^2 \omega_0^n}{\omega'^n} \right) - \Psi(x) = -\log \left(\frac{a \det(h_{ij})_{1 \leq i, j \leq n-1}}{\det(g'_{ij})_{1 \leq i, j \leq n}} \right) (x) - \Psi(x) + O(\|S(x)\|), \quad (11)$$

for x near D .

Since $a^{-1} \det(g'_{ij})_{1 \leq i, j \leq n}|_D$ is a well-defined volume form on D , it makes sense to set

$$f_0(x) = -\log \left(\frac{a \det(h_{ij})_{1 \leq i, j \leq n-1} e^\Psi}{\det(g'_{ij})_{1 \leq i, j \leq n}} \right) (z', 0) + O(\|S(x)\|),$$

for $x = (z', z_n)$. Hence, $\lim_{x \rightarrow D} f_0(x)$ is a constant if and only if $\frac{a \det(h_{ij})_{1 \leq i, j \leq n-1} e^\Psi}{\det(g'_{ij})_{1 \leq i, j \leq n}}(z', 0)$ is constant. In other words, $\lim_{x \rightarrow D} f_0(x)$ is a constant if and only if

$$\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \Psi = -\log \det(h_{ij})_{1 \leq i, j \leq n-1} - \log a \det(g'_{ij})_{1 \leq i, j \leq n}.$$

Since $\Omega = \text{Ric}(g') - \tilde{\omega} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \Psi$, $\lim_{x \rightarrow D} f_0(x)$ is a constant if and only if $\text{Ric}(g_D) = \Omega|_D$. \square

An appropriate choice of Ψ allows us to assume in the sequel that $f_0(x)$ converges uniformly to zero as $x \rightarrow D$.

The function $f_0(x)$ was only defined for x near D , but we can extend it smoothly to be zero along D since $\|S\|^2 \omega_0^n$ is a well-defined volume form over all \overline{M} . Hence, there exists a $\delta_0 > 0$ such that, in the neighborhood $V_0 := \{x \in \overline{M}; \|S(x)\| < \delta_0\}$, f_0 can be written as

$$f_0 = S \cdot u_1 + \overline{S} \cdot \overline{u}_1,$$

where u_1 is a C^∞ local section in $\Gamma(V_0, L_D^{-1})$.

Our goal now would be to construct a function ϕ_1 of the form $S \cdot \theta_1 + \overline{S} \cdot \overline{\theta}_1$, so that the corresponding $f_{\phi_1} = f_1$ vanishes at order 2 along D , and then proceed inductively to higher order. Unfortunately, there is an obstruction to higher order approximation that lies in the kernel of the laplacian on L_D^{-1} restricted to D . In order to deal with this difficulty, one must introduce $(-\log \|S\|^2)$ -terms in the expansion of ϕ_1 , as pointed out in [F] and [LM] where the similar problem of finding expansions for the solutions of the Monge-Ampère equation on a strictly pseudoconvex domain was treated. Further details can be found below.

Following the techniques in [TY2], we are going to construct inductively a sequence of hermitian metrics $\{\|\cdot\|_m\}_{m>0}$ on L_D such that, for any $m > 0$, there exists a $\delta_m > 0$ satisfying:

1. The corresponding Kähler form ω_m associated to $\|\cdot\|_m$ (as defined in (7)) is positive definite in $V_m := \{x \in \overline{M}; \|S(x)\| < \delta_m\}$.
2. The function f_m associated to ω_m (as in the Definition 2.1) can be expanded in V_m as

$$f_m = \sum_{k \geq m+1} \sum_{\ell=0}^{\ell_k} u_{k\ell} (-\log \|S\|_m^2)^\ell, \quad (12)$$

where $u_{k\ell}$ are smooth functions on \overline{V}_m that vanish to order k on D . In particular the function $u_{k\ell}$ can be written as

$$u_{k\ell} = \sum_{i+j=k} S^i \overline{S}^j \theta_{ij} + S^j \overline{S}^i \overline{\theta}_{ij},$$

for $\theta_{ij} \in \Gamma(V_m, L_D^{-i} \otimes \overline{L}_D^{-j})$.

We define $||\cdot||_0 = ||\cdot||$, and it is clear that $||\cdot||_0$ satisfies the Conditions 1 and 2 above. Now we proceed on the inductive step: assuming the existence of $||\cdot||_m$, we construct $||\cdot||_{m+1}$. The next lemma gives a relation between f_m and f_ϕ , where $||\cdot||_\phi = e^{-\phi/2}||\cdot||_m$, and f_ϕ is associated to a smooth function ϕ on V_m of the form

$$\phi = \left(\sum_{i+j=k} S^i \bar{S}^j \theta_{ij} + S^j \bar{S}^i \bar{\theta}_{ij} \right) (-\log ||S||_m^2)^k, \quad \text{for } k \geq m+1.$$

Lemma 2.2 *Let $f_\phi(x)$ be defined as in Definition 2.1, associated to*

$$\omega_\phi = \frac{\sqrt{-1}}{2\pi} \frac{n}{n+1} n^{1/n} \partial \bar{\partial} (-\log ||S||_m^2 \cdot \phi)^{\frac{n+1}{n}}.$$

Similarly let f_m be associated to ω_m . Then

$$\begin{aligned} f_\phi = f_m + nm\phi + \frac{k\phi}{(-\log ||S||_m^2)} & \left(\frac{k-1}{(-\log ||S||_m^2)} + (m-1) \right) + \\ & + (-\log ||S||_m^2)^{k-1} \sum_{i+j=m+1} \left\{ ij \left(S^i \bar{S}^j \theta_{ij} + S^j \bar{S}^i \bar{\theta}_{ij} \right) + \right. \\ & + (-\log ||S||_m^2) \left[-2(n+1)j \left(S^i \bar{S}^j \theta_{ij} + S^j \bar{S}^i \bar{\theta}_{ij} \right) + S^i \bar{S}^j \square_m \theta_{ij} + S^j \bar{S}^i \bar{\square}_m \bar{\theta}_{ij} \right] \Big\} \\ & + \sum_{k' \geq m+2} \sum_{\ell=0}^{\ell_{k'}} u_{k'\ell} (-\log ||S||_m^2)^\ell \end{aligned} \quad (13)$$

where $\square_m = \text{tr}_{\omega_D} (\overline{D_m} D_m)$ is the Laplacian of the bundle $L_D^{-1} \otimes L_D^{-j}$ on D with respect to the hermitian metric $||\cdot||_m$, and the functions $u_{k'\ell}$ decay as $O(||S||^{k'})$.

The proof of this lemma will be postponed to the next section so that we can now proceed to our inductive construction.

Proof of Proposition 2.1: We want to find a function ϕ such that $||\cdot||_{m+1}^2 = e^{-\phi/2}||\cdot||_m^2$ satisfies the Conditions 1 and 2, ie, we need to eliminate the terms $\sum_{\ell=0}^{\ell_{m+1}} u_{m+1,\ell} (-\log ||S||_m^2)^\ell$ from the expansion of f_m . Each of the $u_{m+1,\ell}$, $0 \leq \ell \leq m+1$ will be eliminated successively, as follows.

Step 1: Write $u_{m+1,\ell_{m+1}}$ as

$$u_{m+1,\ell_{m+1}} = \sum_{i+j=m+1} S^i \bar{S}^j (v_{ij} + v'_{ij}) + S^j \bar{S}^i (\bar{v}_{ij} + \bar{v}'_{ij}),$$

where $v'_{ij}|_D \in \text{Ker}(\square_m + n(m+1) - 1 - 2(n+1)j)$ and $v_{ij}|_D$ is perpendicular to that kernel. If there is some i, j ($i+j = m+1$) such that $v'_{ij}|_D \neq 0$, we use Lemma 2.2 with $k = \ell_{m+1} + 1$ and $\theta_{ij} = \frac{v'_{ij} \cdot k(m+1)}{-1-ij}$. Note that the constant $\frac{k(m+1)}{-1-ij}$ was chosen so as to eliminate the kernel term from the expression of $u_{m+1,\ell_{m+1}}$.

Now Lemma 2.2 implies

$$\begin{aligned} f'_m := f_\phi = & \left(\sum_{i+j=m+1} S^i \bar{S}^j v_{ij} + S^j \bar{S}^i \bar{v}_{ij} \right) (-\log ||S||_m^2)^{\ell_{m+1}} + \\ & + \sum_{\ell=0}^{\ell_{m+1}-1} u_{m+1,\ell} (-\log ||S||_m^2)^\ell + \sum_{k' \geq m+1} \sum_{\ell=0}^{\ell_{k'}} u_{k'\ell} (-\log ||S||_m^2)^\ell. \end{aligned} \quad (14)$$

After Step 1, we can assume (by replacing f_m by f'_m in (14)) that f_m has an expansion of the form

$$u_{m+1,\ell_{m+1}} = \sum_{i+j=m+1} S^i \bar{S}^j (v_{ij}) + S^j \bar{S}^i (\bar{v}_{ij}).$$

Step 2: Now we can solve

$$(\square_m + n(m+1) - 1 - 2(n+1)j) \theta_{ij} = v_{ij}|_D \text{ on } D,$$

for $\theta_{ij} \in \Gamma(V_m, L_D^{-i} \otimes \overline{L}_D^{-j})$. Next let us extend θ_{ij} to \overline{M} , and then apply again Lemma 2.2 with $k = \ell_{m+1}$ and θ_{ij} as above. The new f_m will have an expansion of the form

$$f_m = \sum_{\ell=0}^{\ell_{m+1}-1} u_{m+1,\ell} (-\log \|S\|_m^2)^\ell + O(\|S\|^{m+2}).$$

By repeating Steps 1 and 2 above, we are able to eliminate all the terms $\sum_{\ell=0}^{\ell_{m+1}-1} u_{m+1,\ell} (-\log \|S\|_m^2)^\ell$ from the expansion of f_m . Finally, let ϕ_m be the sum of all functions used in steps 1 and 2, and define the new metric $\|\cdot\|_{m+1}$ by letting $\|\cdot\|_{m+1} = e^{-\phi/2} \|\cdot\|_m$. Clearly the resulting metric satisfies Conditions 1 and 2 of (12). This completes the proof of the proposition. \square

3 Proof of Lemma 2.2

This section is entirely devoted to proving Lemma (2.2) therefore completing the inductive construction of the metrics $\|\cdot\|_m$.

According to Definition 2.1, we have $f_\phi(x) = f_m - \log \left(\frac{\omega_\phi^n}{\omega_m^n} \right)$. Hence, we just need to compute the quotient $\frac{\omega_\phi^n}{\omega_m^n}$.

Denote by D_m (resp. D_ϕ) the covariant derivative of the metric $\|\cdot\|_m$ (resp. $\|\cdot\|_\phi$). Similarly, let $\tilde{\omega}_m$ and $\tilde{\omega}_\phi$ denote the corresponding curvature forms. The following relations are well-known:

$$\begin{aligned} D_\phi S &= D_m S - S \partial \phi \\ \tilde{\omega}_\phi &= \tilde{\omega}_m + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi \end{aligned} \quad (15)$$

For simplicity, set $\alpha_m = (-n \log \|S\|_m^2)$ and $\alpha_\phi = (-n \log \|S\|_\phi^2) = \alpha_m + n\phi$. Then,

$$\omega_m^n = \alpha_m \tilde{\omega}_m^{n-1} \wedge \left(\tilde{\omega}_m + \frac{n\sqrt{-1}}{2\pi\alpha_m} \frac{D_m S \wedge \overline{D_m S}}{|S|^2} \right). \quad (16)$$

$$\begin{aligned} \omega_\phi^n &= \alpha_\phi \tilde{\omega}_\phi^{n-1} \wedge \left(\tilde{\omega}_\phi + \frac{n\sqrt{-1}}{2\pi\alpha_\phi} \frac{D_\phi S \wedge \overline{D_\phi S}}{|S|^2} \right) = (\alpha_m + n\phi) \left(\tilde{\omega}_m + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi \right)^{n-1} \wedge \\ &\wedge \left\{ \left(\tilde{\omega}_m + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi \right) + \frac{n\sqrt{-1}}{2\pi\alpha_\phi} \left[\frac{D_m S \wedge \overline{D_m S}}{|S|^2} - \partial \phi \wedge \frac{\overline{D_m S}}{\overline{S}} - \bar{\partial} \phi \wedge \frac{D_m S}{S} + \partial \phi \wedge \bar{\partial} \phi \right] \right\}. \end{aligned} \quad (17)$$

Some calculations using the definition of ϕ lead to

$$\begin{aligned} \partial \phi &= \sum_{i+j=m+1} (-\log \|S\|_m^2)^k \left(D_m S^i \overline{S}^j \theta_{ij} + D_m S^j \overline{S}^i \bar{\theta}_{ij} + S^i \overline{S}^j D_m \theta_{ij} + S^j \overline{S}^i D_m \bar{\theta}_{ij} \right) + \\ &+ k(-\log \|S\|_m^2)^{k-1} (S^i \overline{S}^j \theta_{ij} + S^j \overline{S}^i \bar{\theta}_{ij}) \left(-\frac{D_m S}{S} \right). \end{aligned} \quad (18)$$

We will also need the expression for $\partial\bar{\partial}\phi$. After some computations using (18), it follows that

$$\begin{aligned} \partial\bar{\partial}\phi = & (-\log\|S\|_m^2)^k \sum_{i+j=m+1} \left\{ ij(S^i\bar{S}^j\theta_{ij} + S^j\bar{S}^i\bar{\theta}_{ij}) \left(\frac{D_m S \wedge \overline{D_m S}}{|S|^2} \right) + \right. \\ & + (jS^i\bar{S}^j D_m \theta_{ij} + iS^j\bar{S}^i D_m \bar{\theta}_{ij}) \wedge \frac{\overline{D_m S}}{S} + \frac{D_m S}{S} \wedge (iS^i\bar{S}^j \overline{D_m \theta_{ij}} + jS^j\bar{S}^i \overline{D_m \bar{\theta}_{ij}}) + \\ & + (S^i\bar{S}^j D_m \overline{D_m \theta_{ij}} + S^j\bar{S}^i D_m \overline{D_m \bar{\theta}_{ij}}) + k(-\log\|S\|_m^2)^{-1} \left[(S^i\bar{S}^j \overline{D_m \theta_{ij}} + S^j\bar{S}^i \overline{D_m \bar{\theta}_{ij}}) \wedge \frac{D_m S}{S} + \right. \\ & \left. \left. (S^i\bar{S}^j D_m \theta_{ij} + S^j\bar{S}^i D_m \bar{\theta}_{ij}) \wedge \frac{\overline{D_m S}}{S} \right] \right\} + \left(\frac{k(m+1)\phi}{(-\log\|S\|_m^2)} + \frac{k(k-1)\phi}{(-\log\|S\|_m^2)^2} \right) \frac{D_m S \wedge \overline{D_m S}}{|S|^2}. \quad (19) \end{aligned}$$

We can therefore conclude from a simple analysis of (19) that

$$\|S\|_m^2 (\partial\bar{\partial}\phi)^\ell \wedge \tilde{\omega}_m^{n-\ell} = \tilde{\omega}_m^n O(\|S\|_m^{2m+2}) \text{ for } \ell \geq 2. \quad (20)$$

The above ingredients are going to be needed in the proof of Lemma 2.2.

Proof of Lemma 2.2: Recall that we only need to compute the quotient $\frac{\omega_\phi^n}{\omega_m^n}$. Formulas (16) and (17) then provide

$$\begin{aligned} \frac{\omega_\phi^n}{\omega_m^n} = & \frac{\|S\|_m^2 \alpha_\phi}{\alpha_m(\|S\|_m^2 + 1/\alpha_m\|D_m S\|_m^2) \tilde{\omega}_m^n} \left\{ \left(\tilde{\omega}_m + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\phi \right)^{n-1} \wedge \left\{ \left(\tilde{\omega}_m + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\phi \right) + \right. \right. \\ & \left. \left. + \frac{n\sqrt{-1}}{2\pi\alpha_\phi} \left[\frac{D_m S \wedge \overline{D_m S}}{|S|^2} - \partial\phi \wedge \frac{\overline{D_m S}}{S} - \bar{\partial}\phi \wedge \frac{D_m S}{S} + \partial\phi \wedge \bar{\partial}\phi \right] \right\} \right\} \\ = & \frac{\|S\|_m^2 \alpha_\phi}{(\alpha_m\|S\|_m^2 + \|D_m S\|_m^2) \tilde{\omega}_m^n} \cdot \{ [\tilde{\omega}_m^{n-1} + (n-1)\tilde{\omega}_m^{n-2} \wedge \partial\bar{\partial}\phi] \wedge \\ \left[\tilde{\omega}_m + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\phi + \frac{n\sqrt{-1}}{2\pi\alpha_\phi} \left(\frac{D_m S \wedge \overline{D_m S}}{|S|^2} - \partial\phi \wedge \frac{\overline{D_m S}}{S} - \bar{\partial}\phi \wedge \frac{D_m S}{S} + \partial\phi \wedge \bar{\partial}\phi \right) \right] \} + O(\|S\|_m^{m+2}) \quad (21) \end{aligned}$$

Direct calculations using (18) and its analogous formula for $\bar{\partial}\phi$ lead to (further details can be found in [S])

$$\partial\phi \wedge \frac{\overline{D_m S}}{S} + \bar{\partial}\phi \wedge \frac{D_m S}{S} = \left((m+1)\phi + \frac{2k\phi}{\log\|S\|_m^2} \right) \frac{D_m S \wedge \overline{D_m S}}{|S|^2} + O(\|S\|_m^{m+1}). \quad (22)$$

Thus it follows that

$$\begin{aligned} \frac{\omega_\phi^n}{\omega_m^n} = & \frac{\|S\|_m^2 \alpha_\phi}{\alpha_m(\|S\|_m^2 + 1/\alpha_m\|D_m S\|_m^2) \tilde{\omega}_m^n} \cdot \left\{ [\tilde{\omega}_m^{n-1} + (n-1)\tilde{\omega}_m^{n-2} \partial\bar{\partial}\phi] \wedge \left[\tilde{\omega}_m + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\phi + \right. \right. \\ & \left. \left. \frac{n\sqrt{-1}}{2\pi\alpha_\phi} \left(1 - (m+1)\phi - \frac{2k\phi}{\log\|S\|_m^2} \right) \frac{D_m S \wedge \overline{D_m S}}{|S|^2} + \partial\phi \wedge \bar{\partial}\phi \right] \right\} + O(\|S\|_m^{m+2}). \quad (23) \end{aligned}$$

On the other hand, (22) implies that the summand with $\partial\phi \wedge \bar{\partial}\phi$ in (23) can be bounded by a multiple of $\|S\|_m^{m+3} \tilde{\omega}_m^n$. So, we obtain

$$\begin{aligned} \frac{\omega_\phi^n}{\omega_m^n} = & \frac{\|S\|_m^2 \alpha_\phi}{\alpha_m(\|S\|_m^2 + 1/\alpha_m\|D_m S\|_m^2) \tilde{\omega}_m^n} \cdot \\ & \left\{ \tilde{\omega}_m^{n-1} \wedge \left[\tilde{\omega}_m + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\phi + \frac{n\sqrt{-1}}{2\pi\alpha_\phi} \left(1 - (m+1)\phi - \frac{2k\phi}{\log\|S\|_m^2} \right) \frac{D_m S \wedge \overline{D_m S}}{|S|^2} \right] + \right. \\ & \left. + (n-1)\tilde{\omega}_m^{n-2} \wedge \left(\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\phi \right) \wedge \left(\tilde{\omega}_m + \frac{n\sqrt{-1}}{2\pi\alpha_\phi} \frac{D_m S \wedge \overline{D_m S}}{|S|^2} \right) \right\} + O(\|S\|_m^{m+2}). \quad (24) \end{aligned}$$

Also, the definitions of α_ϕ and α_m give

$$\alpha_\phi \|S\|_m^2 \tilde{\omega}_m^{n-1} \wedge \left(\tilde{\omega}_m + \frac{n\sqrt{-1}}{2\pi\alpha_\phi} \frac{D_m S \wedge \overline{D_m S}}{|S|^2} \right) = \|S\|_m^2 \tilde{\omega}_m^n \left(\alpha_m + \frac{\|D_m S\|_m^2}{\|S\|_m^2} \right) + O(\|S\|_m^{m+2}) \tilde{\omega}_m^n. \quad (25)$$

Therefore,

$$\begin{aligned} \frac{\omega_\phi^n}{\omega_m^n} &= 1 + \frac{\|S\|_m^2 \alpha_\phi}{\alpha_m (\|S\|_m^2 + 1/\alpha_m \|D_m S\|_m^2) \tilde{\omega}_m^n} \cdot \\ &\quad \left\{ n \tilde{\omega}_m^{n-1} \wedge \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi \right) - \phi \left((m+1) + \frac{2k}{(-\log \|S\|_m^2)} \right) \cdot \left[\tilde{\omega}_m^{n-1} \wedge \frac{n\sqrt{-1}}{2\pi\alpha_\phi} \frac{D_m S \wedge \overline{D_m S}}{|S|^2} \right] + \right. \\ &\quad \left. + (n-1) \tilde{\omega}_m^{n-2} \wedge \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi \right) \wedge \left(\frac{n\sqrt{-1}}{2\pi\alpha_\phi} \frac{D_m S \wedge \overline{D_m S}}{|S|^2} \right) \right\} + O(\|S\|_m^{m+2}) = \\ &= 1 - \left((m+1) + \frac{2k}{(-\log \|S\|_m^2)} \right) \phi + \frac{\|S\|_m^2 \alpha_\phi}{\alpha_m (\|S\|_m^2 + 1/\alpha_m \|D_m S\|_m^2) \tilde{\omega}_m^n} \cdot \\ &\quad \left\{ n \tilde{\omega}_m^{n-1} \wedge \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi \right) + (n-1) \tilde{\omega}_m^{n-2} \wedge \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi \right) \wedge \left(\frac{n\sqrt{-1}}{2\pi\alpha_\phi} \frac{D_m S \wedge \overline{D_m S}}{|S|^2} \right) \right\} + O(\|S\|_m^{m+2}) = \\ &= 1 - \left((m+1) + \frac{2k}{(-\log \|S\|_m^2)} \right) \phi + \frac{\|S\|_m^2 \alpha_\phi}{\alpha_m (\|S\|_m^2 + 1/\alpha_m \|D_m S\|_m^2) \tilde{\omega}_m^n} \cdot \\ &\quad \cdot \left\{ (-\log \|S\|_m^2)^k \sum_{i+j=m+1} [ij(S^i \bar{S}^j \theta_{ij} + S^j \bar{S}^i \bar{\theta}_{ij})] \left(\tilde{\omega}_m^{n-1} \wedge \left(\frac{n\sqrt{-1}}{2\pi\alpha_\phi} \frac{D_m S \wedge \overline{D_m S}}{|S|^2} \right) \right) + \right. \\ &\quad \left. + (n-1) \tilde{\omega}_m^{n-2} \wedge \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi \right) \wedge \left(\frac{n\sqrt{-1}}{2\pi\alpha_\phi} \frac{D_m S \wedge \overline{D_m S}}{|S|^2} \right) + \right. \\ &\quad \left. + \phi \left(\frac{n(m+1)k}{(-\log \|S\|_m^2)} + \frac{n(k-1)k}{(-\log \|S\|_m^2)^2} \right) \left(\tilde{\omega}_m^{n-1} \wedge \frac{n\sqrt{-1}}{2\pi\alpha_\phi} \frac{D_m S \wedge \overline{D_m S}}{|S|^2} \right) \right\} + O(\|S\|_m^{m+2}). \quad (26) \end{aligned}$$

The last term of (26) can be simplified by using (25), yielding

$$\begin{aligned} \frac{\omega_\phi^n}{\omega_m^n} &= 1 + \left(-(m+1) + \frac{k(m-1)}{(-\log \|S\|_m^2)} + \frac{(k-1)k}{(-\log \|S\|_m^2)^2} \right) \phi + \frac{\alpha_\phi \|S\|_m^2}{\alpha_m (\|S\|_m^2 + 1/\alpha_m \|D_m S\|_m^2) \tilde{\omega}_m^n} \cdot \\ &\quad \cdot \left\{ (-\log \|S\|_m^2)^k \sum_{i+j=m+1} [ij(S^i \bar{S}^j \theta_{ij} + S^j \bar{S}^i \bar{\theta}_{ij})] \tilde{\omega}_m^{n-1} \wedge \left(\frac{n\sqrt{-1}}{2\pi\alpha_\phi} \frac{D_m S \wedge \overline{D_m S}}{|S|^2} \right) + \right. \\ &\quad \left. + (n-1) \tilde{\omega}_m^{n-2} \wedge \left(\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi \right) \wedge \left(\frac{n\sqrt{-1}}{2\pi\alpha_\phi} \frac{D_m S \wedge \overline{D_m S}}{|S|^2} \right) \right\}. \quad (27) \end{aligned}$$

Now observe that the relations

$$\begin{aligned} \overline{D_m} D_m S^j &= -D_m \overline{D_m} S^j + j S^j \tilde{\omega}_m \\ \overline{D_m} D_m \theta_{ij} &= -D_m \overline{D_m} \theta_{ij} - (i-j) \theta_{ij} \tilde{\omega}_m \end{aligned} \quad (28)$$

imply that

$$\begin{aligned} \sum_{i+j=m+1} \left(S^i \bar{S}^j D_m \overline{D_m} \theta_{ij} + S^j \bar{S}^i D_m \overline{D_m} \bar{\theta}_{ij} \right) &= \\ \sum_{i+j=m+1} \left\{ \left(S^i \bar{S}^j \overline{D_m} D_m \theta_{ij} + S^j \bar{S}^i \overline{D_m} D_m \bar{\theta}_{ij} \right) - \left(S^i \bar{S}^j \theta_{ij} - S^j \bar{S}^i \bar{\theta}_{ij} \right) (i-j) \tilde{\omega}_m \right\}. \quad (29) \end{aligned}$$

Hence,

$$\begin{aligned} \frac{\omega_\phi^n}{\omega_m^n} &= 1 + \left(-(m+1) + \frac{k(m-1)}{(-\log \|S\|_m^2)} + \frac{k(k-1)}{(-\log \|S\|_m^2)^2} \right) \phi + \frac{\alpha_\phi \|S\|_m^2}{\alpha_m (\|S\|_m^2 + 1/\alpha_m \|D_m S\|_m^2) \tilde{\omega}_m^n} \\ &\quad \left\{ (-\log \|S\|_m^2)^k \sum_{i+j=m+1} [\alpha_\phi i j (S^i \bar{S}^j \theta_{ij} + S^j \bar{S}^i \bar{\theta}_{ij})] \tilde{\omega}_m^{n-1} \wedge \left(\frac{n\sqrt{-1} D_m S \wedge \overline{D_m S}}{2\pi\alpha_\phi |S|^2} \right) + \right. \\ &\quad \left. + (n-1) \tilde{\omega}_m^{n-2} \wedge \left(\frac{n\sqrt{-1} D_m S \wedge \overline{D_m S}}{2\pi\alpha_\phi |S|^2} \right) \wedge \left[- \left(S^i \bar{S}^j \overline{D_m} D_m \theta_{ij} + S^j \bar{S}^i \overline{D_m} D_m \bar{\theta}_{ij} \right) \right. \right. \\ &\quad \left. \left. + \left(S^i \bar{S}^j \theta_{ij} + S^j \bar{S}^i \bar{\theta}_{ij} \right) (i-j) \tilde{\omega}_m \right] \right\} + O(\|S\|_m^{m+2}). \quad (30) \end{aligned}$$

Notice that we can replace the term involving α_ϕ from the last formula by the analogous term involving α_m , since the function ϕ is assumed to be of order $O(\|S\|_m^{m+1})$. This implies that the residual term of this substitution will lie in the term $O(\|S\|_m^{m+2})$. Therefore, we conclude

$$\begin{aligned} \frac{\omega_\phi^n}{\omega_m^n} &= 1 + \left(-(m+1) + \frac{k(m-1)}{(\log \|S\|_m^2)} + \frac{k(k-1)}{(\log \|S\|_m^2)^2} \right) \phi - \\ &\quad - (-\log \|S\|_m^2)^k \sum_{i+j=m+1} \left(S^i \bar{S}^j \square_m \theta_{ij} + S^j \bar{S}^i \square_m \bar{\theta}_{ij} \right) + (-\log \|S\|_m^2)^k \cdot \\ &\quad \cdot \sum_{i+j=m+1} \alpha_m i j + (n-1)((m+1)-2j) \left(S^i \bar{S}^j \theta_{ij} + S^j \bar{S}^i \bar{\theta}_{ij} \right) + O(\|S\|_m^{m+2}) = \\ &\quad = 1 + n(m+1)\phi + \frac{k}{(-\log \|S\|_m^2)} \left(\frac{k-1}{(-\log \|S\|_m^2)} + (m-1) \right) + \\ &\quad + \sum_{i+j=m+1} \left(S^i \bar{S}^j \square_m \theta_{ij} + S^j \bar{S}^i \square_m \bar{\theta}_{ij} \right) + 2(n-1)(-\log \|S\|_m^2)^k \sum_{i+j=m+1} j \left(S^i \bar{S}^j \theta_{ij} + S^j \bar{S}^i \bar{\theta}_{ij} \right) + \\ &\quad + (\log \|S\|_m^2)^{k-1} \sum_{i+j=m+1} i j \left(S^i \bar{S}^j \theta_{ij} - S^j \bar{S}^i \bar{\theta}_{ij} \right) + O(\|S\|_m^{m+2}). \quad (31) \end{aligned}$$

Finally,

$$\begin{aligned} f_\phi &= f_m - \log \left(\frac{\omega_\phi^n}{\omega_m^n} \right) = nm\phi + \frac{k\phi}{(-\log \|S\|_m^2)} \left(\frac{k-1}{(-\log \|S\|_m^2)} - (m-1) \right) + \\ &\quad + (-\log \|S\|_m^2)^k \sum_{i+j=m+1} \left[\left(S^i \bar{S}^j \square_m \theta_{ij} + S^j \bar{S}^i \square_m \bar{\theta}_{ij} \right) + 2(n+1)j \left(S^i \bar{S}^j \theta_{ij} - S^j \bar{S}^i \bar{\theta}_{ij} \right) \right] + \\ &\quad + (-\log \|S\|_m^2)^{k-1} \sum_{i+j=m+1} i j \left(S^i \bar{S}^j \theta_{ij} - S^j \bar{S}^i \bar{\theta}_{ij} \right) + O(\|S\|_m^{m+2}), \quad (32) \end{aligned}$$

which proves the lemma. The inductive construction of the metrics $\|\cdot\|_m$ is also completed. \square

4 Complete Kähler Metrics on M

In this section we shall finish the proof of Theorem 1.1. In particular it is going to be necessary to consider the asymptotic behavior of the Riemann curvature tensor.

For each $m \geq 1$, consider the function f_m constructed in Section 2. For this choice, let the corresponding

$$\omega_m = \frac{\sqrt{-1}}{2\pi(n+1)} \partial \bar{\partial} (-n \log \|S\|_m^2)^{\frac{n+1}{n}}$$

define a $(1,1)$ -form on M . If δ_m is sufficiently small, ω_m is positive definite on $V_m = \{\|S(x)\| \leq \delta_m\}$, and defines a Kähler metric g_m .

Lemma 4.1 *The Kähler manifolds $(V_m, \partial V_m, g_m)$ are all complete, equivalent to each other near D , and for each $m > 0$, the function*

$$\rho = \frac{2}{n+1}(-n \log \|S\|_m^2)^{\frac{n+1}{2n}}$$

is equivalent to any distance function from a fixed point in V_m near D .

Proof: Fix $m > 0$. We have

$$|\nabla_m \rho|_{g_m}^2 = \frac{\sqrt{-1}}{2\pi} \frac{\partial \rho \wedge \bar{\partial} \rho \wedge \omega_m^{n-1}}{\omega_m^n}.$$

Since

$$\partial \rho = n^{\frac{n+1}{2n}}(-\log \|S\|_m^2)^{\frac{1-n}{2n}} \frac{D_m S}{S}$$

it follows that

$$\frac{\sqrt{-1}}{2\pi} \partial \rho \wedge \bar{\partial} \rho \wedge \omega_m^{n-1} = (-n \log \|S\|_m^2)^{\frac{1-n}{2n}} (-n \log \|S\|_m^2)^{\frac{n-1}{2n}} \tilde{\omega}_m^{n-1} \wedge \frac{\sqrt{-1}}{2\pi} \frac{D_m S \wedge \overline{D_m S}}{|S|^2}. \quad (33)$$

Therefore, we conclude that

$$|\nabla_m \rho|_{g_m}^2 = \frac{1}{n} \frac{\tilde{\omega}_m^{n-1} \wedge \frac{n\sqrt{-1}}{2\pi} \frac{D_m S \wedge \overline{D_m S}}{|S|^2}}{(-n \log \|S\|_m^2) \tilde{\omega}_m^n + \tilde{\omega}_m^{n-1} \wedge \frac{n\sqrt{-1}}{2\pi} \frac{D_m S \wedge \overline{D_m S}}{|S|^2}} = \frac{1}{n} \frac{\|D_m S\|_m^2}{(-n \log \|S\|_m^2) + \|D_m S\|_m^2}. \quad (34)$$

Now, recall that $\|D_m S\|_m^2$ is never zero, and that $\lim_{\|S\|_m \rightarrow 0} (-\log \|S\|_m^2 \cdot \|S\|_m^2) = 0$. Hence

$$|\nabla_m \rho|_{g_m}^2 \xrightarrow{\|S\|_m \rightarrow 0} \frac{1}{n},$$

proving that ρ is equivalent to any distance function from the boundary near D .

Also, since $\rho \rightarrow \infty$ when $x \rightarrow D$, the Kähler manifold $(V_m, \partial V_m, g_m)$ is complete.

We claim that all the metrics g_m are equivalent near D . To check the claim, note first that each $\tilde{\omega}_m$ is the curvature form of the metric $\|\cdot\|_m$, hence, for every $m, \ell \in \mathbb{N}$, $\tilde{\omega}_m$ is equivalent to $\tilde{\omega}_\ell$ near D . The claim then follows from (8), that relates the expressions for ω_m and $\tilde{\omega}_m$.

Finally here is a remark about the volume growth of $(V_m, \partial V_m, g_m)$: since ω_m^n is equivalent to $\tilde{\omega}_m^n (-n \log \|S\|_m^2)$, it suffices to consider the integral

$$\int_{\|S(x)\|_m \geq e^{-1/2n\rho} \frac{2n}{n+1}} (-n \log \|S\|_m^2) \tilde{\omega}_m^n,$$

which is of order $\rho^{\frac{2n}{n+1}}$. □

In the sequel we are going to carry out the estimates of the Riemann curvature tensor $R(g_m)$ corresponding to the metric g_m which are involved in the statement of Theorem (1.1). Let us begin with the following lemma:

Lemma 4.2 *Let $(V_m, \partial V_m, g_m)$ be complete, Kähler manifolds with boundary defined as in Lemma 4.1. Then the norm of $R(g_m)$ with respect to the metric g_m decays at the order of at least $(-n \log \|S\|_m^2)^{-\frac{1}{n}}$ near D .*

Proof: We shall prove the statement in local coordinates, as follows. There exists a finite covering U_t of D in \overline{M} such that for each t , there is a local uniformization $\Pi_t : \tilde{U}_t \rightarrow U_t$ such that $\Pi_t^{-1}(D)$ is smooth in \tilde{U}_t . The covering U_t can, in addition, be chosen so that given a local coordinate system (z_1, \dots, z_n) in \tilde{U}_t , with $S = z_n$ and $z' = (z_1, \dots, z_{n-1})$ defining coordinates along D , we have

$$\begin{aligned} \sum_{i,j,k,l=1}^n R(\Pi_t^*(g_m))_{i\bar{j}k\bar{l}}(z', z_n) \xi^i \bar{\xi}^{\bar{j}} \xi^k \bar{\xi}^{\bar{l}} &= \\ &= (-n \log |z_n|^2)^{1/n} \sum_{i,j,k,l=1}^n R(\Pi_t^*(g_D|_{\Pi_t^{-1}(D)}))_{i\bar{j}k\bar{l}} \xi^i \bar{\xi}^{\bar{j}} \xi^k \bar{\xi}^{\bar{l}} + O((-n \log |z_n|^2)^{-1/n}), \end{aligned} \quad (35)$$

for every g_m -unit vector (ξ^1, \dots, ξ^n) , where g_D is the Kähler metric defined by the restriction of the curvature form $\tilde{\omega}$ to the divisor.

Without loss of generality, we assume that $U_t \cap \overline{M}$ is smooth.

For every $x \in U_t \cap M$, consider local coordinates (z_1, \dots, z_n) for a neighborhood of x which satisfy the conditions below:

- The defining section S of the divisor is given by z_n .
- The curvature form $\tilde{\omega}_m$ of $||\cdot||_m$ is represented in the mentioned coordinates by the tensor $(h_{i\bar{j}})$ where $(h_{i\bar{j}})$ satisfies

$$h_{i\bar{j}}(x) = \delta_{ij}; \quad \frac{\partial h_{i\bar{j}}}{\partial z_k}(x) = 0 \quad \text{if } j < n; \quad \frac{\partial h_{i\bar{j}}}{\partial \bar{z}_l}(x) = 0 \quad \text{if } i < n;$$

- The hermitian metric $||\cdot||_m$ is represented by a positive function a with $a(x) = 1$, $da(x) = 0$ and $d(\frac{\partial a}{\partial z_k})(x) = 0$.

In order to simplify notation, let us write $B = B(|z_n|) = (-n \log |z_n|^2)$, and let us drop the subscripts for the metric g_m to be denoted by g from now on.

Formula (8) implies that

$$g^{i\bar{j}}(x) = \begin{cases} O(B^{-1/n}) & \text{if } i = j \text{ and } i < n \\ o(B^{-1/n}) & \text{if } i \neq j \text{ and } i, j < n \\ O(|z_n|^2 B^{-1/n}) & \text{if } i = j = n, \end{cases}$$

and computations give (check [S] for further details)

$$\frac{\partial g_{i\bar{j}}}{\partial z_k} = B^{1/n} \left[\frac{\partial h_{i\bar{j}}}{\partial z_k} - \frac{1}{z_n B} \left(\delta_{kn} h_{i\bar{j}} + \delta_{in} h_{k\bar{j}} + \delta_{in} \delta_{jn} \delta_{kn} \left(\frac{n-1}{B} + \frac{1}{|z_n|^2} \right) \right) \right]$$

and

$$\begin{aligned} \frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} = B^{1/n} & \left\{ \frac{\partial^2 h_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} - \frac{1}{\bar{z}_n B} \left(\delta_{ln} \frac{\partial h_{i\bar{j}}}{\partial z_k} + \delta_{jn} \frac{\partial h_{i\bar{l}}}{\partial z_k} \right) - \frac{1}{z_n B} \left(\delta_{kn} \frac{\partial h_{i\bar{j}}}{\partial \bar{z}_l} + \delta_{in} \frac{\partial h_{k\bar{j}}}{\partial \bar{z}_l} \right) + \right. \\ & + \frac{1-n}{|z_n|^2 B^2} (\delta_{ln} (\delta_{kn} h_{i\bar{j}} + \delta_{in} h_{k\bar{j}}) + \delta_{jn} (\delta_{kn} h_{i\bar{l}} + \delta_{in} h_{k\bar{l}})) + \\ & \left. + \frac{\delta_{in} \delta_{jn} \delta_{kn} \delta_{ln}}{|z_n|^4 B^3} (|z_n|^2 (n-1)(1-2n) + (1-n)B + B^2) \right\}. \quad (36) \end{aligned}$$

If (ξ^1, \dots, ξ^n) is a g -unit tangent vector, then

$$\begin{cases} |\xi^i|^2 \leq C B^{-1/n} & \text{if } i < n, \text{ and} \\ |\xi^n|^2 \leq C |z_n|^2 B^{(n-1)/n}, \end{cases}$$

where C is a constant that does not depend neither on the unit vector (ξ^1, \dots, ξ^n) nor on the point $x \in D$. Hence, in local coordinates, we have

$$\begin{aligned} R(g)_{i\bar{j}k\bar{l}}(x)(\xi^i \bar{\xi}^j \xi^k \bar{\xi}^l)(x) &= \left[\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l}(x) + \sum_{u,v=1}^n g^{u\bar{v}}(x) \frac{\partial g_{i\bar{v}}}{\partial z_k}(x) \frac{\partial g_{u\bar{j}}}{\partial \bar{z}_l}(x) \right] (\xi^i \bar{\xi}^j \xi^k \bar{\xi}^l) = \\ &= -B^{1/n} \left\{ \frac{\partial^2 h_{i\bar{j}}}{\partial z_k \partial \bar{z}_l}(x)(\xi^i \bar{\xi}^j \xi^k \bar{\xi}^l) + \frac{4(1-n)|\xi^n|^2}{|z_n|^2 B^2} \left(\sum_{i=1}^n |\xi^i|^2 \right) \right. \\ & \quad \left. + \frac{|\xi^n|^4}{|z_n|^4 B^3} (|z_n|^2 (n-1)(1-2n) + (1-n)B + B^2) \right\} + \\ & \quad + B^{2/n} \sum_{u,v=1}^n g^{u\bar{v}}(x) \left[-\frac{\xi^n}{z_n B} \left(2\xi^i h_{i\bar{v}} + \xi^n \delta_{i\bar{v}} \left(\frac{n-1}{B} + \frac{1}{|z_n|^2} \right) \right) \right. \\ & \quad \left. \cdot \left[-\frac{\bar{\xi}^n}{z_n B} \left(2\bar{\xi}^j h_{u\bar{j}} + \bar{\xi}^n \delta_{u\bar{n}} \left(\frac{n-1}{B} + \frac{1}{|z_n|^2} \right) \right) \right] \right] \quad (37) \end{aligned}$$

Next let us separately bound each of the above terms. By using (4) and our previous choice of local coordinates, we obtain, when z_n approaches zero,

$$\frac{4(1-n)|\xi^n|^2}{|z_n|^2 B^2} \left(\sum_{i=1}^n |\xi^i|^2 \right) \leq \frac{C|z_n|^2 B^{(n-1)/n}}{|z_n|^2 B^2} (B^{-1/n} + |z_n|^2 B^{(n-1)/n}) \leq C B^{-(n+2)/n},$$

where C denotes a uniform constant. Also, we have that

$$\frac{|\xi^n|^4}{|z_n|^4 B^3} (|z_n|^2 (n-1)(1-2n) + (1-n)B + B^2) \leq C B^{-1/n} (B^{-1} + 1) \leq C B^{-1/n}.$$

Now, notice that the expression

$$\frac{\xi^n}{z_n B} \left(2\xi^i h_{i\bar{v}} + \xi^n \delta_{i\bar{v}} \left(\frac{n-1}{B} + \frac{1}{|z_n|^2} \right) \right) \leq C (B^{-(n+2)/2n} + |z_n|^{-1} B^{-1/n}), \quad (38)$$

needs special attention due to the presence of a term involving $|z_n|^{-1}$. However our estimate for

$$g^{u\bar{v}} = B^{-1/n} [(1 - \delta_{un} \delta_{vn}) O(1) + \delta_{un} \delta_{vn} O(|z_n|^2)]$$

shows that this term is compensated by the last term of the above expression. The estimate for the decay of the last term in (37) is analogous to the case discussed above, and will be omitted. For details, please refer to [S].

In conclusion, we have

$$R(g)_{i\bar{j}k\bar{l}}(x)(\xi^i \bar{\xi}^j \xi^k \bar{\xi}^l)(x) = B^{1/n} \frac{\partial^2 h_{i\bar{j}}}{\partial z_k \partial \bar{z}_l}(x)(\xi^i \bar{\xi}^j \xi^k \bar{\xi}^l) + O(B^{-1/n}),$$

which implies (35), and concludes the proof of the lemma. \square

The reader may also notice that Lemma 4.2 completes the proof of Proposition 2.1.

Lemma 4.3 *For the manifold $(V_m, \partial V_m, g_m)$ defined in Lemma 4.1, we have*

$$\|\nabla^k R(g_m)\|_{g_m}(x) = O\left(\rho_m^{-\frac{k+2}{n+1}}(x)\right), \quad (39)$$

where $\rho_m(x)$ is the distance function from a fixed point associated to g_m .

Proof: The proof of this lemma will follow the same idea as the proof of [TY2], Lemma 2.5.

We start by fixing m , and fixing a small $\delta > 0$ such that $V_\delta := \{x \in M; \|S\|_m < \delta\} \subset V_m$. In order to prove the result, we are going to introduce a suitable new coordinate system on V_δ .

Because D is admissible, it follows that the total space of the unit sphere bundle of $L_D|_D$ (with respect to the metric $\|\cdot\|_m$) is a smooth manifold of real dimension $2n+1$, to be denoted by M_1 .

Since L_D is simply the normal bundle of D in \overline{M} , there exists a diffeomorphism

$$\Psi : M_1 \times (0, \delta) \rightarrow V_\delta$$

induced by the exponential map of $(\overline{M}, \|\cdot\|_m)$ along D .

It is also known that the Kähler form of g_m is given by

$$\omega_m = \frac{\sqrt{-1}}{2\pi} \frac{n^{1+1/n}}{n+1} \partial \bar{\partial} (-\log(\|S\|^2 e^{2\phi_m}))^{\frac{n+1}{n}},$$

where ϕ_m is a smooth function on \overline{M} , that can be written as $\sum_{\kappa \geq m} \sum_{\ell=0}^{\ell_\kappa} u_{\kappa\ell} (-\log \|S\|_k^2)^\ell$, where $u_{\kappa\ell}$ are smooth functions on \overline{V}_m that vanish to order κ on D .

Combining the facts above, the pull-back of g_m under Ψ on $M_1 \times (0, \delta)$ is given by

$$\begin{aligned}\Psi^* g_m &= (-n \log(\|S\|^2))^{\frac{1}{n}} g(\|S\|, \|S\| \log(\|S\|)) + \\ &+ (-n \log(\|S\|^2))^{\frac{1-n}{n}} d \left((-n \log(\|S\|^2))^{\frac{1}{n}} \right) h(\|S\|, \|S\| \log(\|S\|)) + \\ &+ (-n \log(\|S\|^2))^{\frac{1-2n}{n}} d \left((-n \log(\|S\|^2))^{\frac{1}{n}} \right)^2 u(\|S\|, \|S\| \log(\|S\|)).\end{aligned}\quad (40)$$

Here $g(\cdot, \cdot)$ (resp. $h(\cdot, \cdot)$, $u(\cdot, \cdot)$) is a C^∞ family of metrics (resp. 1-tensors, functions) on M_1 , such that for each fixed integer $\ell > 0$, there exists a constant K_ℓ that bounds all covariant derivatives (with respect to a fixed metric h on M_1) of $g(t_0, t_1)$ (resp. $h(t_0, t_1)$, $u(t_0, t_1)$) up to order ℓ , for every $t_0 \in [0, \delta]$ and $t_1 \in [0, \delta \log(\delta)]$.

Setting $\rho = (-n \log(\|S\|^2))^{\frac{n+1}{2n}}$, (40) becomes

$$\Psi^* g_m = \rho^{\frac{2}{n+1}} g(\cdot, \cdot) + \rho^{\frac{2(1-n)}{n+1}} d(\rho^{\frac{2}{n+1}}) h(\cdot, \cdot) + \rho^{\frac{2(1-2n)}{n+1}} d(\rho^{\frac{2}{n+1}})^2 u(\cdot, \cdot), \quad (41)$$

and hence we can regard $\Psi^* g_m$ as being a metric defined on $M_1 \times \left((-n \log \delta^2)^{\frac{n+1}{2n}}, \infty \right)$.

It is easy to see that, if g is any Riemannian metric, and $\tilde{g} = \lambda^2 g$ is a rescaling of g , then the norm of the covariant derivatives of the new metric, taken with respect to the new metric \tilde{g} , satisfy $\|\tilde{\nabla}^k R(\tilde{g})\|_{\tilde{g}} = \lambda^{-(k+2)} \|\nabla^k R(g)\|_g$. Therefore, if we define a new metric \tilde{g} on $M_1 \times \left((-n \log \delta^2)^{\frac{n+1}{2n}}, \infty \right)$ by $\tilde{g} = \rho_m^{-\frac{2}{n+1}}(x) \Psi^* g_m$, showing (39) is equivalent to showing that $\|\tilde{\nabla}^k R(\tilde{g})\|_{\tilde{g}}(x) = O(1)$.

But this last statement follows clearly from the assertions on $g(\cdot, \cdot)$, $h(\cdot, \cdot)$ and $u(\cdot, \cdot)$. \square

In the same lines of Lemma 4.3, one can show

Lemma 4.4 *For a fixed m , let f_m be the smooth function constructed in Section 2. Then,*

$$\|\nabla^k f_m\|_{g_m} = O(\|S\|_m^{m+1} \rho_m^{-\frac{k}{n+1}}), \quad (42)$$

where we recall that $\rho_m(\cdot)$ is the distance function associated to the metric g_m .

Proof: We will use the same notation and objects defined on the proof of Lemma 4.3.

Begin by observing that (42) is equivalent to showing that

$$\|\tilde{\nabla}^k f_m\|_{\rho_m^{-\frac{2}{n+1}} \Psi^* g_m}(\Psi^{-1}(x)) = O_k(1)(\|S\|_m^{m+1})(x), \quad (43)$$

where $O_k(1)$ is a quantity bounded by a constant that depends on k , and $\tilde{\nabla}$ is the covariant derivative associated to the metric $\rho_m^{-\frac{2}{n+1}} \Psi^* g_m$ on $M_1 \times (0, \delta)$.

Also, recall that the function f_m satisfies $\text{Ric}(g_m) - \Omega = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} f_m$ on V_δ , for a fixed $\Omega \in c_1(-K_{\overline{M}} \otimes -L_D)$.

Hence, (43) follows clearly from the estimates on the covariant derivatives of the curvature tensor $R(\rho_m^{-\frac{2}{n+1}} \Psi^* g_m)$ given by Lemma 4.3, and the observation on the local expression of the metric $\rho_m^{-\frac{2}{n+1}} \Psi^* g_m$. \square

We are finally able to prove Theorem 1.1.

Proof of Theorem 1.1: In what follows we keep the preceding setting and notations.

Since the divisor D is assumed to be ample in \overline{M} , there exists a hermitian metric $\|\cdot\|'$ on L_D whose curvature form $\tilde{\omega}'$ is positive definite on D .

Fix an integer $k \geq \varepsilon$, and write, for $\varepsilon > 0$,

$$\omega_{g_\varepsilon} = \omega_k + C_\varepsilon \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} (-||S||')^\varepsilon, \quad C_\varepsilon > 0, \quad (44)$$

where ω_k is the Kähler form defined on Section 2. The Kähler form ω_{g_ε} is positive definite on M , and gives rise to a complete Kähler metric g_ε on M .

Let $\delta > 0$ be such that $V_\delta = \{||S(x)|| < \delta\} \subset V_k$. On V_δ , ω_{g_ε} satisfies $\text{Ric}(g_\varepsilon) - \Omega = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} f$, and we want to estimate the decay of f at infinity.

On the other hand, on V_δ , we have $\text{Ric}(g_k) - \Omega = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} f_k$ which, in turn, implies that

$$\begin{aligned} f = f_k - \log \frac{\omega_{g_\varepsilon}^n}{\omega_k^n} &= f_k - \log \left(\frac{w_k^n + C_\varepsilon ||S||'^{(\varepsilon-1)} w_k^{n-1} \wedge \frac{n\sqrt{-1}}{2\pi} (D'S \wedge \bar{D}'\bar{S})}{\omega_k^n} \right) = \\ &= f_k - C_\varepsilon ||S||'^{(\varepsilon-1)} ||D'S||_{g_k}, \end{aligned} \quad (45)$$

where $D'S$ denotes the covariant derivative of the metric $||\cdot||'$. Hence, in order to estimate the decay of f , it suffices to study the decay of $||D'S||_{g_k}$.

In what follows, we will use the notation defined in the proof of Lemma 4.3: recall that Ψ is the map between a tubular neighborhood of D inside \bar{M} induced by the exponential map of $||\cdot||$ along D .

Let the function $\gamma := \Psi^*(||S||')^\varepsilon$ be defined on $M_1 \times \left((-n \log \delta^2)^{\frac{n+1}{2n}}, \infty\right)$. Our goal is to understand the decay of $||D'S||_{g_k}$, which is equivalent to studying the decay of $|\tilde{\nabla} \gamma|_{\rho^{-2/(n+1)} \Psi^* g_k}(\Psi^{-1}(x))$, where $\tilde{\nabla}$ denotes the covariant derivative of the metric $\rho^{-2/(n+1)} \Psi^* g_k$.

Notice that on $M_1 \times \left((-n \log \delta^2)^{\frac{n+1}{2n}}, \infty\right)$, the function γ can be written under the form

$$e^{\tilde{\gamma}(\cdot, \exp\{\rho^{\frac{2n}{n+1}}\})} \exp\left\{\frac{\varepsilon}{n} \rho^{\frac{2n}{n+1}}\right\},$$

where $\tilde{\gamma}$ is a smooth function on $M_1 \times \left((-n \log \delta^2)^{\frac{n+1}{2n}}, \infty\right)$ with all derivatives bounded in terms of a fixed product metric.

Hence, from (41), it follows that

$$|\tilde{\nabla} \gamma|_{\rho^{-2/(n+1)} \Psi^* g_k}(\Psi^{-1}(x)) = O(\exp\left\{\frac{\varepsilon}{n} \rho^{\frac{2n}{n+1}}\right\}), \quad (46)$$

since the curvature tensor of $\rho^{-2/(n+1)} \Psi^* g_k$ is bounded near $\Psi^{-1}(x)$.

Note also that (46) is equivalent to

$$||D'S||_{g_k} = O(||S||^\varepsilon),$$

which shows that the metric g_ε , with corresponding Kähler form ω_{g_ε} defined by (44), satisfies the equation $\text{Ric}(g_\varepsilon) - \Omega = \partial \bar{\partial} f_\varepsilon$, for f_ε a smooth function that decays at least to the order of $O(||S||^\varepsilon)$.

To close the proof of Theorem (1.1), it only remains to note that the curvature estimates for the new metric g_ε follow easily from the estimates on the curvature tensor $R(g_m)$, described in Lemma 4.2. \square

5 Asymptotics of the Monge-Ampère Equation on M

This last section is intended to provide the proof of Theorem 1.2.

Let (M, g) be a complete Kähler manifold, with Kähler form ω . Consider the following Monge-Ampère equation on M :

$$\begin{cases} \left(\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u\right)^n = e^f \omega^n, \\ \omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u > 0, \end{cases} \quad u \in C^\infty(M, \mathbb{R}), \quad (47)$$

where f is a given smooth function satisfying the integrability condition

$$\int_M (e^f - 1) \omega^n = 0. \quad (48)$$

Whenever u is a solution to (47), the $(1, 1)$ -form $\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u$ satisfies $\text{Ric}(\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u) = f$. So, in order to define metrics with prescribed Ricci curvature, it is enough to determine a solution to (47).

In [TY1], Tian and Yau proved that (47) has, in fact, solutions modulo assuming certain conditions on the volume growth of g as well as on the decay of f at infinity. For the convenience of the reader, we state here their main result.

Theorem 5.1 (Tian, Yau, [TY1]) *Let (M, g) be a complete Kähler manifold, satisfying:*

- *The sectional curvature of the metric g is bounded by a constant K .*
- *$\text{Vol}_g(B_R(x_0)) \leq CR^2$ for all $R > 0$ and $\text{Vol}_g(B_1(x_0)) \geq C^{-1}(1 + \rho(x))^{-\beta}$, for a constant β , where Vol_g denotes the volume associated to the metric g , $B_R(x_0)$ is the geodesic ball of radius R about a fixed point $x_0 \in M$, and $\rho(x)$ denotes the distance (with respect to g) from x_0 to x .*
- *There are positive numbers $r > 0, r_1 > r_2 > 0$ such that for every $x \in M$, there exists a holomorphic map $\phi_x : \mathcal{U}_x \subset (\mathbb{C}^n, 0) \rightarrow B_r(x)$ such that $\phi_x(0) = x$. Here one has $B_{r_2} \subset \mathcal{U}_x \subset B_{r_1}$, where $B_r := \{z \in \mathbb{C}^n; |z| \leq r\}$. Furthermore $\phi_x^* g$ is a Kähler metric in \mathcal{U}_x whose metric tensor has derivatives up to order 2 bounded and $1/2$ -Hölder-continuously bounded.*

Let f be a smooth function, satisfying the integrability condition (48) and such that

$$\sup_M \{|\nabla_g f|, |\Delta_g f|\} \leq C \quad |f(x)| \leq C(1 + \rho(x))^{-N}, \quad (49)$$

for some constant C , for all x in M , where $N \geq 4 + 2\beta$.

Then there exists a bounded, smooth solution u for (47), such that $\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u$ defines a complete Kähler metric equivalent to g .

An interesting question addressed to by Tian and Yau in the same paper is that whether or not the resulting metric is asymptotically as close to g as possible modulo assuming further conditions on the decay of f . We provide an answer to this problem in the remainder of this paper.

We are interested in studying the Monge-Ampère equation (47) for the Kähler manifold $(M, \omega_{g_\varepsilon})$ constructed in Section 4. More precisely, given $\varepsilon > 0$, we want to understand the asymptotic behavior of a solution u to the problem

$$\begin{cases} \left(\omega_{g_\varepsilon} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u \right)^n = e^{f_{g_\varepsilon}} \omega_{g_\varepsilon}^n, \\ \omega_{g_\varepsilon} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u > 0, \end{cases} \quad u \in C^\infty(M, \mathbb{R}). \quad (50)$$

To guarantee the existence of a solution to (50), we need to check that the function f_{g_ε} (defined on Theorem 1.1) satisfies the integrability condition (48).

Lemma 5.1 *There exists a number $\lambda > 0$ such that, by replacing ϕ by $\phi + \lambda$ in Definition (2.1) of f_ϕ , we have*

$$\int_M (e^{f_{g_\varepsilon}} - 1) \omega_{g_\varepsilon}^n = 0. \quad (51)$$

Proof: Recall the definition of ω_{g_ε} :

$$\omega_{g_\varepsilon} = \omega_\phi + C_\varepsilon \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} (-||S||')^{2\varepsilon}, \quad (52)$$

where $\omega_\phi = \frac{\sqrt{-1}}{2\pi} \frac{n^{1+1/n}}{n+1} \partial\bar{\partial}(-\log \|S\|_\phi^2)^{\frac{n+1}{n}}$. We choose ϕ (as in Section 2) so that the corresponding f_ϕ decays faster than $O(\|S\|^\epsilon)$.

A direct computation using integration by parts shows that $\int_M \omega_{g_\varepsilon}^n - \omega_\phi^n = 0$. Also, the definitions of ω_{g_ε} and ω_ϕ imply that $e^{f_{g_\varepsilon}} \omega_{g_\varepsilon}^n = e^{f_\phi} \omega_\phi^n$. Therefore, $\int_M (e^{f_{g_\varepsilon}} - 1) \omega_{g_\varepsilon}^n = \int_M (e^{f_\phi} - 1) \omega_\phi^n$. On the other hand, Definition 2.1 gives

$$e^{f_\phi} \omega_\phi^n = \frac{e^{-\Psi} \omega'^n}{\|S\|^2}. \quad (53)$$

Notice that the function Ψ remains unchanged if we replace ϕ by $\phi + \lambda$, since $\tilde{\omega}_\phi = \tilde{\omega}_{\phi+\lambda}$. Therefore, the right-hand side of (53) is invariant under the transformation $\phi \mapsto \phi + \lambda$.

On the other hand, a straightforward computation using (7) shows that

$$\omega_{\phi+\lambda}^n = \left(\frac{\sqrt{-1}}{2\pi} \frac{n^{1+1/n}}{n+1} \partial\bar{\partial}(-\log \|S\|_{\phi+\lambda}^2)^{\frac{n+1}{n}} \right) = \left(\frac{\sqrt{-1}}{2\pi} \frac{n^{1+1/n}}{n+1} \partial\bar{\partial}(-\log \|S\|_\phi^2)^{\frac{n+1}{n}} \right) - n\lambda \tilde{\omega}_\phi^n, \quad (54)$$

where $\tilde{\omega}_\phi^n$ is the curvature form of the hermitian metric $\|\cdot\|_\phi$.

Therefore, by redefining f_ϕ by $f_\phi = -\log \|S\|^2 - \log \frac{\omega_{\phi+\lambda}^n}{\omega'^n} - \Psi$, we have that

$$\int_M (e^{f_\phi} - 1) \omega_{\phi+\lambda}^n = \int_M \left(\frac{e^{-\Psi} \omega'^n}{\|S\|^2} - \omega_\phi^n \right) - n\lambda \int_M \tilde{\omega}_\phi^n. \quad (55)$$

Since the first integral in the above expression is finite and independent on λ , we can choose the number λ so as to make the right-hand side of (55) equal to *zero*. This establishes the lemma. \square

The previous lemma shows that each f_{g_ε} satisfies the conditions in the existence theorem of Tian and Yau. Also, the estimates on the decay of the Riemann curvature tensor (Lemma 4.2) and the observation on the volume growth of the metric g_ε (see the remark after Lemma 4.1) show that (M, g_ε) is a complete Kähler manifold in which Theorem 5.1 can be applied.

Therefore, for each $\varepsilon > 0$, there exists a bounded and smooth solution u_ε to the problem (50). Our goal now is to understand the asymptotic behavior of u_ε .

Denote by ω_Ω the Kähler form on M given by Theorem 5.1, when we use g_ε (given by Theorem 1.1) as the ambient metric:

$$\omega_\Omega = \omega_{g_\varepsilon} + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} u_\varepsilon.$$

Clearly, it suffices to prove the asymptotic assertions on u_ε for a small tubular neighborhood of D in \overline{M} . Recall from Theorem 1.1 that on $V_\varepsilon \setminus D$,

$$\omega_{g_\varepsilon} = \omega_m + C_\varepsilon \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} (\|S\|')^{2\varepsilon},$$

for some $m \geq \varepsilon$ fixed.

Since ω_m and ω_{g_ε} are cohomologous, there exists a function u_m such that we can write, in $V_m \setminus D$,

$$\omega_\Omega = \omega_m + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} u_m. \quad (56)$$

On the other hand, if f_m is the function defined by (2.1), (56) implies that u_m satisfies

$$\left(\omega_m + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} u_m \right)^n = e^{f_m} \omega_m^n \quad \text{on } V_m \setminus D, \quad (57)$$

where we remind the reader that $|f_m|_{g_m}$ is of order of $O(\|S\|_m^m)$.

Lemma 5.2 *On the neighborhood $V_m \setminus D = \{0 < \|S\|_m < \delta_m\}$, we have*

$$\begin{aligned} & \left\{ \omega_m + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \left(C \left[S^i \bar{S}^j \theta_{ij} + \bar{S}^i S^j \bar{\theta}_{ij} \right] (-n \log(\|S\|_m^2))^k \right) \right\}^n = \\ & = \omega_m^n \left[1 + C(-n \log(\|S\|_m^2))^{k-\frac{n+1}{n}} \left\{ ij(-n \log(\|S\|_m^2))^2 \left[S^i \bar{S}^j \theta_{ij} + \bar{S}^i S^j \bar{\theta}_{ij} \right] - \right. \right. \\ & \quad \left. \left. - (-n \log(\|S\|_m^2)) \left[(k(i+j) + j(n-1)) S^i \bar{S}^j \theta_{ij} + (k(i+j) + i(n-1)) \bar{S}^i S^j \bar{\theta}_{ij} \right] \right. \right. \\ & \quad \left. \left. + k(k-n) \right\} + O(\|S\|_m^{i+j+1}) \right], \quad (58) \end{aligned}$$

where θ_{ij} is a C^∞ local section of $L_D^{-i} \otimes \bar{L}_D^j$ on V_m .

Proof: In order to simplify the notation, define $B = (-n \log(\|S\|_m^2))$. Computations (that can be found in detail in [S]) lead to

$$\begin{aligned} & \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \left(CB^k \left[S^i \bar{S}^j \theta_{ij} + \bar{S}^i S^j \bar{\theta}_{ij} \right] \right) = \tilde{\omega}_m \left\{ CB^{k-1} \left[(-jB+k) S^i \bar{S}^j \theta_{ij} \right] + \left[(-iB+k) \bar{S}^i S^j \bar{\theta}_{ij} \right] \right\} + \\ & + \frac{\sqrt{-1}}{2\pi} \frac{D_m S \wedge \bar{D}_m \bar{S}}{|S|^2} \left\{ CB^{k-2} \left[S^i \bar{S}^j \theta_{ij} + \bar{S}^i S^j \bar{\theta}_{ij} \right] \left[ijF^2 - k(i+j)F + k(k-1) \right] \right\} + \\ & + CB^{k-1} \frac{\sqrt{-1}}{2\pi} \left[(jB-k) S^i \bar{S}^j D_m \theta_{ij} + (iB-k) \bar{S}^i S^j D_m \bar{\theta}_{ij} \right] \wedge \frac{\bar{D}_m \bar{S}}{\bar{S}} + \\ & + CB^{k-1} \frac{\sqrt{-1}}{2\pi} \frac{D_m S}{S} \wedge \left[(iB-k) S^i \bar{S}^j \bar{D}_m \theta_{ij} + (jB-k) \bar{S}^i S^j \bar{D}_m \bar{\theta}_{ij} \right] + \\ & + CB^k \frac{\sqrt{-1}}{2\pi} \left[S^i \bar{S}^j D_m \bar{D}_m \theta_{ij} + \bar{S}^i S^j D_m \bar{D}_m \bar{\theta}_{ij} \right], \quad (59) \end{aligned}$$

where D_m stands for the covariant derivative with respect to the hermitian metric $\|\cdot\|_m$, and where $\tilde{\omega}_m$ is its corresponding curvature form.

From (8), we have

$$\omega_m = (-n \log(\|S\|_m^2))^{\frac{1}{n}} \tilde{\omega}_m + (-n \log(\|S\|_m^2))^{\frac{1-n}{n}} \frac{\sqrt{-1}}{2\pi} \frac{D_m S \wedge \bar{D}_m \bar{S}}{|S|^2}.$$

Thus

$$\begin{aligned} & \left[\omega_m + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \left(CB^k \left[S^i \bar{S}^j \theta_{ij} + \bar{S}^i S^j \bar{\theta}_{ij} \right] \right) \right]^n = \left[a \tilde{\omega}_m + b \frac{\sqrt{-1}}{2\pi} \frac{D_m S \wedge \bar{D}_m \bar{S}}{|S|^2} + \right. \\ & \left. \frac{\sqrt{-1}}{2\pi} (c_1 D_m \theta_{ij} + c_2 D_m \bar{\theta}_{ij}) \wedge \frac{\bar{D}_m \bar{S}}{\bar{S}} + \frac{\sqrt{-1}}{2\pi} \frac{D_m S}{S} \wedge (d_1 \bar{D}_m \theta_{ij} + d_2 \bar{D}_m \bar{\theta}_{ij}) + e \frac{\sqrt{-1}}{2\pi} \bar{D}_m D_m \theta_{ij} \right]^n, \quad (60) \end{aligned}$$

where

$$\begin{aligned} a &= B^{\frac{1}{n}} \left[1 - CB^{k-\frac{n+1}{n}} \left[(jB-k) S^i \bar{S}^j \theta_{ij} (iB-k) \bar{S}^i S^j \bar{\theta}_{ij} + \right] \right] \\ b &= B^{\frac{1-n}{n}} \left[1 + CB^{k-\frac{n+1}{n}} \left[S^i \bar{S}^j \theta_{ij} + \bar{S}^i S^j \bar{\theta}_{ij} \right] \left[ijF^2 - k(i+j)F + k(k-1) \right] \right] \\ c_1 &= CS^i \bar{S}^j B^{k-1} [jB-k], & c_2 &= C \bar{S}^i S^j B^{k-1} [iB-k] \\ d_1 &= CS^i \bar{S}^j B^{k-1} [iB-k], & d_2 &= C \bar{S}^i S^j B^{k-1} [jB-k] \\ e &= CS^i \bar{S}^j B^k. \end{aligned} \quad (61)$$

Now, we proceed on estimating each term on (60).

$$a^n \tilde{\omega}_m^n = B^{\frac{1}{n}} \left[1 - CnB^{k-\frac{n+1}{n}} \left[(jB-k) S^i \bar{S}^j \theta_{ij} + (iB-k) \bar{S}^i S^j \bar{\theta}_{ij} \right] + O(\|S\|_m^{i+j+1}) \right] \tilde{\omega}_m^n. \quad (62)$$

Also

$$\begin{aligned}
na^{n-1}b &= \left[1 - C(n-1)B^{k-\frac{n+1}{n}} \left[(jB-k)S^i\bar{S}^j\theta_{ij} + (iB-k)\bar{S}^iS^j\bar{\theta}_{ij} \right] \right] \cdot \\
&\quad \cdot \left[1 + CB^{k-\frac{n+1}{n}} \left[S^i\bar{S}^j\theta_{ij} + \bar{S}^iS^j\bar{\theta}_{ij} \right] \left[ijB^2 - k(i+j)B + k(k-1) \right] \right] = \\
&\quad = 1 + CB^{k-\frac{n+1}{n}} \left\{ ijB^2 \left[S^i\bar{S}^j\theta_{ij} + \bar{S}^iS^j\bar{\theta}_{ij} \right] - \right. \\
&\quad \left. -B \left[(k(i+j) + j(n-1)) S^i\bar{S}^j\theta_{ij} + (k(i+j) + i(n-1)) \bar{S}^iS^j\bar{\theta}_{ij} \right] + k(k-n) \right\} + O(\|S\|_m^{i+j+1}). \quad (63)
\end{aligned}$$

The expressions for the other terms are analogous, and will henceforth be omitted.

From (7), we deduce that

$$\tilde{\omega}_m^n = \omega_m^n \frac{\|S\|_m^2 B^{-1}}{\|S\|_m^2 + B^{-1}\|D_m S\|_m^2},$$

and since

$$\frac{\|S\|_m^2}{\|S\|_m^2 + B^{-1}\|D_m S\|_m^2} = \frac{\|S\|_m^2 B^{-1}}{\|D_m S\|_m^2} \left(\frac{1}{1 + \frac{B\|S\|_m^2}{\|D_m S\|_m^2}} \right) = O(\|S\|_m^2 B^{-1}),$$

all the terms in (60) will decay at the order of at least $O(\|S\|_m^{i+j+1})$, with the exception of the term (63), which will be written as:

$$\begin{aligned}
a^{n-1}b \left\{ \tilde{\omega}_m^{n-1} \wedge \frac{n\sqrt{-1}}{2\pi} \frac{D_m S \wedge \overline{D_m S}}{|S|^2} \right\} &= a^{n-1}b \frac{\|D_m S\|_m^2}{\|S\|_m^2} \tilde{\omega}_m^n = \\
&= a^{n-1}b \left(\frac{\|D_m S\|_m^2}{\|S\|_m^2} \right) \frac{\omega_m^n \|S\|_m^2 B^{-1}}{\|S\|_m^2 + B^{-1}\|D_m S\|_m^2} = \omega_m^n \left[1 + CB^{k-\frac{n+1}{n}} \left\{ ijB^2 \left[S^i\bar{S}^j\theta_{ij} + \bar{S}^iS^j\bar{\theta}_{ij} \right] - \right. \right. \\
&\quad \left. \left. -B \left[(k(i+j) + j(n-1)) S^i\bar{S}^j\theta_{ij} + (k(i+j) + i(n-1)) \bar{S}^iS^j\bar{\theta}_{ij} \right] + k(k-n) \right\} + O(\|S\|_m^{i+j+1}) \right]. \quad (64)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left[\omega_m + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \left(CS^i\bar{S}^j\theta_{ij}B^k \right) \right]^n &= \omega_m^n \left[1 + CB^{k-\frac{n+1}{n}} \left\{ ijB^2 \left[S^i\bar{S}^j\theta_{ij} + \bar{S}^iS^j\bar{\theta}_{ij} \right] - \right. \right. \\
&\quad \left. \left. -B \left[(k(i+j) + j(n-1)) S^i\bar{S}^j\theta_{ij} + (k(i+j) + i(n-1)) \bar{S}^iS^j\bar{\theta}_{ij} \right] + k(k-n) \right\} + O(\|S\|_m^{i+j+1}) \right], \quad (65)
\end{aligned}$$

completing the proof of the lemma. \square

Proposition 5.1 *Let u_m be a solution to the Monge-Ampère equation (57). If $u_m(x)$ converges uniformly to zero as x approaches the divisor, then there exists a constant $C = C(m)$ such that*

$$|u_m(x)| \leq C\|S\|_m^{m+1} \quad \text{on } V_m \setminus D. \quad (66)$$

Proof: It suffices to prove (66) in a neighborhood of D . Apply Lemma 5.2 for $i = m+2$ and $j = -1$, and choose the section θ_{ij} so that the function $S^i\bar{S}^j\theta_{ij} + \bar{S}^iS^j\bar{\theta}_{ij}$ is positive on $V_m \setminus D$. Note that there is, in fact, a C^∞ -section θ_{ij} satisfying this condition. Indeed, a local section on a trivializing coordinate can clearly be constructed by means of a bump function. In particular we can consider finitely many local sections as above such that the union of their supports covers the all of D . Since the positivity condition is naturally respected by the cocycle relations arising from the change of coordinates, the desired section θ_{ij} can simply be obtained by adding these local sections.

With the above choices, we have

$$\begin{aligned}
\left[\omega_m + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} C \left[S^i\bar{S}^j\theta_{ij} + \bar{S}^iS^j\bar{\theta}_{ij} \right] (-n \log(\|S\|_m^2))^k \right]^n &= \\
&= \omega_m^n \left[1 - C(m+2)(-n \log(\|S\|_m^2))^{k+\frac{n-1}{n}} \left\{ [1 + o(1)] \left[S^i\bar{S}^j\theta_{ij} + \bar{S}^iS^j\bar{\theta}_{ij} \right] \right\} + O(\|S\|_m^{m+2}) \right]. \quad (67)
\end{aligned}$$

On the other hand,

$$e^{f_m} \omega_m^n = [1 + O(\|S\|_m^{m+1})] \omega_m^n \quad \text{on } V_m \setminus D. \quad (68)$$

More precisely, we can write on $V_m \setminus D$

$$e^{f_m} \omega_m^n = [1 + \sum_{\ell=0}^{\ell_{m+1}} \{S^i \bar{S}^j \theta_{ij} + \bar{S}^i S^j \bar{\theta}_{ij}\} (-\log \|S\|_m^2)^\ell + O(\|S\|_m^{m+2})] \omega_m^n, \quad (69)$$

for sections $\theta_{ij} \in \Gamma(V_m \setminus D, L_D^{-i} \otimes \bar{L}_D^{-j})$.

Let $\varepsilon > 0$, and define $C_i = \frac{C'_i}{\varepsilon}$, where $C'_1 := \sup_{x \in V_m \setminus D} (|u_m| + 1)$, and $C'_2 = -C'_1$. Then, for all $x \in V_m \setminus D$ verifying

$$\left(S^{m+2} \bar{S}^{-1} \theta_{m+2,-1} + \bar{S}^{m+2} S^{-1} \bar{\theta}_{m+2,-1} (-n \log(\|S\|_m^2))^{\ell_{m+1}} \right) (x) = \varepsilon, \quad (70)$$

it follows that

$$C_1 \left(S^{m+2} \bar{S}^{-1} \theta_{m+2,-1} + \bar{S}^{m+2} S^{-1} \bar{\theta}_{m+2,-1} (-n \log(\|S\|_m^2))^{\ell_{m+1}} \right) (x) > |u_m(x)|.$$

Furthermore, if ε is sufficiently small, then on the subset

$$\{x \in V_m \setminus D; \left(S^{m+2} \bar{S}^{-1} \theta_{m+2,-1} + \bar{S}^{m+2} S^{-1} \bar{\theta}_{m+2,-1} (-n \log(\|S\|_m^2))^{\ell_{m+1}} \right) (x) \leq \varepsilon\},$$

we have (for $i = m+2$ and $j = -1$)

$$\begin{aligned} \left[\omega_m + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} C_1 \left[S^i \bar{S}^j \theta_{ij} + \bar{S}^i S^j \bar{\theta}_{ij} \right] (-n \log(\|S\|_m^2))^{\ell_{m+1} - \frac{n-1}{n}} \right]^n &\leq e^{f_m} \omega_m^n, \quad \text{and} \\ \left[\omega_m + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} C_2 \left[S^i \bar{S}^j \theta_{ij} + \bar{S}^i S^j \bar{\theta}_{ij} \right] (-n \log(\|S\|_m^2))^{\ell_{m+1} - \frac{n-1}{n}} \right]^n &\geq e^{f_m} \omega_m^n. \end{aligned}$$

Finally, by using the hypothesis on the uniform vanishing of u_m on D , the proposition follows from the maximum principle for the complex Monge-Ampère operator: we obtain the following bound (write $\ell = \ell_{m+1} - \frac{n-1}{n}$)

$$C_2 \left[S^i \bar{S}^j \theta_{ij} + \bar{S}^i S^j \bar{\theta}_{ij} \right] (-n \log(\|S\|_m^2))^\ell \leq u_m \leq C_1 \left[S^i \bar{S}^j \theta_{ij} + \bar{S}^i S^j \bar{\theta}_{ij} \right] (-n \log(\|S\|_m^2))^\ell \quad (71)$$

on the neighborhood given in (70), which completes the proof of the proposition. \square

Now, let us deal with another step in the proof of Theorem 1.2, which consists of showing that the solution to the Monge-Ampère equation (57) actually converges uniformly to zero.

Proposition 5.2 *For a fixed $m \geq 2$, let u_m be a solution to (57). Then $u_m(x)$ converges uniformly to zero as x approaches the divisor D .*

Proof: In [TY1], the solution u_m to the Monge-Ampère equation (47) is obtained as the uniform limit, as ε goes to zero, of solutions $u_{m,\varepsilon}$ of the perturbed Monge-Ampère equations

$$\begin{cases} \left(\omega_m + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u \right)^n = e^{f_m + \varepsilon u} \omega_m^n, \\ \omega_m + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u > 0, \end{cases} \quad u \in C^\infty(M, \mathbb{R}). \quad (72)$$

On the neighborhood V_m , Lemma (5.2) applied for $i = 2$, $j = -1$ and $k = 0$, gives

$$\begin{aligned} \left\{ \omega_m + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \left(C \left[S^2 \bar{S}^{-1} \theta_{2,-1} + \bar{S}^2 S^{-1} \bar{\theta}_{2,-1} \right] \right) \right\}^n &= \\ &= \omega_m^n \left[1 - 2C (-n \log(\|S\|_m^2))^{-\frac{n+1}{n}} \left\{ (-n \log(\|S\|_m^2))^2 \left[S^2 \bar{S}^{-1} \theta_{2,-1} + \bar{S}^2 S^{-1} \bar{\theta}_{2,-1} \right] - \right. \right. \\ &\quad \left. \left. - (-n \log(\|S\|_m^2)) \left[(1-n) S^2 \bar{S}^{-1} \theta_{2,-1} + 2(n-1) \bar{S}^2 S^{-1} \bar{\theta}_{2,-1} \right] \right\} + O(\|S\|_m^2) \right]. \quad (73) \end{aligned}$$

Again, we can choose appropriate local C^∞ -sections $\theta_{2,-1}$ such that

$$\left[S^2 \bar{S}^{-1} \theta_{2,-1} + \bar{S}^2 S^{-1} \bar{\theta}_{2,-1} \right]$$

is a positive function on a neighborhood of the divisor, and use this function as a uniform barrier to the sequence of solutions $\{u_{m,\varepsilon}\}$.

Note that $e^{f_m + \varepsilon u_{m,\varepsilon}} = 1 + O(\|S\|_m)$. Hence, as in the proof of Lemma 5.2, we can define, for a fixed $\delta > 0$, $C_i = \frac{C'_i}{\delta}$, where $C'_1 := \sup_{x \in V_m \setminus D} (|u_{m,\varepsilon}| + 1)$ and $C'_2 = -C'_1$. A priori, C'_i could depend on ε , but it turns out (see [TY1] for details) that $\sup_M |u_{m,\varepsilon}|$ can be bounded uniformly by a constant independent on ε . Then, for all $x \in V_m \setminus D$ such that

$$\left(S^2 \bar{S}^{-1} \theta_{2,-1} + \bar{S}^2 S^{-1} \bar{\theta}_{2,-1} \right) (x) = \delta,$$

we have that

$$C_1 \left(S^2 \bar{S}^{-1} \theta_{2,-1} + \bar{S}^2 S^{-1} \bar{\theta}_{2,-1} \right) (x) > |u_{m,\varepsilon(x)}|.$$

In addition, in the neighborhood $\{x \in V_m \setminus D; \left(S^2 \bar{S}^{-1} \theta_{2,-1} + \bar{S}^2 S^{-1} \bar{\theta}_{2,-1} \right) (x) \leq \delta\}$, for a fixed δ sufficiently small, we have

$$\left[\omega_m + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} C_1 \left[S^2 \bar{S}^{-1} \theta_{2,-1} + \bar{S}^2 S^{-1} \bar{\theta}_{2,-1} \right] \right]^n \leq e^{f_m + \varepsilon C_1 \left[S^2 \bar{S}^{-1} \theta_{2,-1} + \bar{S}^2 S^{-1} \bar{\theta}_{2,-1} \right]} \omega_m^n$$

and

$$\left[\omega_m + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} C_2 \left[S^2 \bar{S}^{-1} \theta_{2,-1} + \bar{S}^2 S^{-1} \bar{\theta}_{2,-1} \right] \right]^n \geq e^{f_m + \varepsilon C_2 \left[S^2 \bar{S}^{-1} \theta_{2,-1} + \bar{S}^2 S^{-1} \bar{\theta}_{2,-1} \right]} \omega_m^n.$$

Since $u_{m,\varepsilon}$ vanishes at D (see [CY1]), we can apply the maximum principle to conclude that there exists a C independent of ε such that, near D ,

$$-C \left[S^2 \bar{S}^{-1} \theta_{2,-1} + \bar{S}^2 S^{-1} \bar{\theta}_{2,-1} \right] \leq u_{m,\varepsilon} \leq C \left[S^2 \bar{S}^{-1} \theta_{2,-1} + \bar{S}^2 S^{-1} \bar{\theta}_{2,-1} \right]$$

Now, since the neighborhood $\{x \in V_m \setminus D; \left(S^2 \bar{S}^{-1} \theta_{2,-1} + \bar{S}^2 S^{-1} \bar{\theta}_{2,-1} \right) (x) \leq \delta\}$ is a fixed set, independent of ε , and the constant C above is also independent of ε , we can pass to the limit when ε goes to zero, obtaining the claim. \square

We can still provide further information about the decay of the covariant derivatives of the solution u_m to (57).

Proposition 5.3 *For a fixed $m > 1$, let u_m be a solution to (57). Then, there exists $C = C(k, m)$ such that, for all $x \in V_m \setminus D$,*

$$|\nabla^k u_m|_{g_m}(x) \leq C \|S\|_m^m(x) \rho_m^{-\frac{k+2}{n+1}}. \quad (74)$$

Proof: The strategy of this proof will be induction on k , and the use of Moser Iteration Technique.

Recall that Lemma 4.3 gives

$$\|\nabla^k R(g_m)\|_{g_m}(x) = O(\rho^{-\frac{k+2}{n+1}})$$

for all $x \in V_m \setminus D$.

In order to use Moser Iteration, we will need to localize on a small ball around a point $x \in V_m \setminus D$. Let us define a new rescaled Kähler metric on $B_R(x, g_m)$ by $\tilde{g} = R^{-2} g_m$, where $R = \rho_m^{\frac{1}{n+1}}(x)$. Note that $B_1(x, \tilde{g}) = B_R(x, g_m)$.

Then,

$$\|\tilde{\nabla}^k R(\tilde{g})\|_{\tilde{g}}(x) = \|\nabla^k R(g_m)\|_{g_m}(x) R^{k+2},$$

and hence $\sup\{|\tilde{\nabla}^k R(\tilde{g})|_{\tilde{g}}(x); x \in B_1(x, \tilde{g})\} \leq C$.

By writing $\tilde{u} = R^{-2}u_m$, we get that \tilde{u} satisfies

$$\left(\omega_{\tilde{g}} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \tilde{u}\right)^n = e^{f_m} \omega_{\tilde{g}}^n \quad \text{on } B_1(x, \tilde{g}), \quad (75)$$

and from Propositions 5.2 and 5.1, we also have that $\sup\{|\tilde{u}|; x \in B_1(x, \tilde{g})\} \leq C \|S\|_m^m(x)$.

Lemma 4.4 also tells us that $|\nabla^k f_m|_{g_m}(x) = O(\|S\|_m^{m+1}(x) \rho_m^{-\frac{k}{n+1}}(x))$, hence there exists a constant $C = C(k, m)$ such that $\sup\{|\nabla^k f_m|_{g_m}; x \in B_1(x, \tilde{g})\} \leq C \|S\|_m^m(x)$.

Now, we have all the ingredients to start the inductive proof. Since $|\nabla^k u_m|_{g_m}(x) = R^{-k+2} |\nabla^k \tilde{u}|_{\tilde{g}}(x)$, it suffices to prove that there exists a constant C such that

$$|\nabla^k \tilde{u}|_{\tilde{g}}(x) \leq C \|S\|_m^m(x). \quad (76)$$

We will proceed by induction on k . First, consider the case $k = 1$. By the second order estimate in [Y] (see also [TY1]), we have that the Kähler metric which solves the Monge-Ampère equation is equivalent to the chosen representative of the fixed Kähler class, *ie*, there exists a constant $C = C_m$ such that

$$0 \leq \omega_m + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u_m \leq C \omega_m.$$

This implies that

$$0 \leq \omega_{\tilde{g}} + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \tilde{u} \leq C \omega_{\tilde{g}} \quad \text{on } B_1(x, \tilde{g}).$$

Let η be a cut-off function that vanishes outside the unit ball, and is identically one on $B_{1/2}(x, \tilde{g})$. By multiplying both sides of (75) by $\eta^2 \tilde{u}$ and integrating by parts, we obtain the L^2 -estimate of $|\nabla \tilde{u}|$:

$$\int_{B_{1/2}(x, \tilde{g})} |\nabla \tilde{u}|_{\tilde{g}}^2 \omega_{\tilde{g}}^n \leq C \int_{B_1(x, \tilde{g})} |e^{f_m} - 1| |\tilde{u}| \omega_{\tilde{g}}^n \leq \|S\|_m^{2m}(x). \quad (77)$$

Now, differentiating (75) with respect to z_k , $1 \leq k \leq n$, we obtain

$$\tilde{g}^{i\bar{j}} \left(\frac{\partial \tilde{u}}{\partial z_k} \right)_{i\bar{j}} = (e^{f_m} - 1) \tilde{g}^{i\bar{j}} \left(\frac{\partial \tilde{g}_{i\bar{j}}}{\partial z_k} \right)_{i\bar{j}} + \frac{\partial f_m}{\partial z_k} e^{f_m} + O(|\nabla^2 \tilde{u}|), \quad (78)$$

where $O(|\nabla^2 \tilde{u}|)$ denotes the terms that can be bounded by $|\nabla^2 \tilde{u}|$.

Our estimates on the L^2 -norm of $|\nabla \tilde{u}|$ on the ball $B_{1/2}(x, \tilde{g})$ allow us to apply Theorem 8.17, [GT] (the Moser Iteration Method) to conclude that

$$|\nabla \tilde{u}|_{\tilde{g}}(x) \leq C \|S\|_m^m(x). \quad (79)$$

Recall that x was a point chosen arbitrarily, so this estimate holds for all $x \in V_m \setminus D$.

Now, the next step is to multiply both sides of (79) by $\eta^2 \frac{\partial \tilde{u}}{\partial z_k}$ (for η an adequate cut-off function defined on $B_1(x, \tilde{g})$) and integrate by parts, in order to obtain the estimate on the L^2 -estimate on the norm of $\nabla^2 \tilde{u}$:

$$\int_{B_{1/2}(x, \tilde{g})} |\nabla^2 \tilde{u}|_{\tilde{g}}^2 \omega_{\tilde{g}}^n \leq C \|S\|_m^{2m}(x). \quad (80)$$

Since it is analogous to our previous step, we will omit further explanations. We note that the pointwise estimate on $|\nabla \tilde{u}|$ allow us to obtain the L^2 -estimate on $|\nabla^2 \tilde{u}|$. This completes the first step on the induction (for $k = 1$).

Now, assume that the following is true for all $j \leq k - 1$:

$$|\nabla^j \tilde{u}|_{\tilde{g}}(x) \leq C \|S\|_m^m(x), \quad \text{and} \quad (81)$$

$$\int_{B_{1/2}(x, \tilde{g})} |\nabla^k \tilde{u}|_{\tilde{g}}^2 \omega_{\tilde{g}}^n \leq C \|S\|_m^{2m}(x). \quad (82)$$

Differentiating (75) k times, we obtain

$$\tilde{g}^{i\bar{j}} \left(\frac{\partial^k \tilde{u}}{\partial z^{i_1} \dots \partial z^{i_k}} \right)_{i\bar{j}} = \frac{\partial^k f_m}{\partial z^{i_1} \dots \partial z^{i_k}} e^{f_m} + O(\|S\|_m^m(x)). \quad (83)$$

Our estimates on the decay of $|\nabla^k f_m|_{\tilde{g}}$ (given by Lemma 4.4) allow us to use again Moser Iteration to conclude that

$$|\nabla^k \tilde{u}|_{\tilde{g}}(x) \leq C \|S\|_m^m(x), \quad (84)$$

completing the proof of the Proposition. \square

Proof of Theorem 1.2: It follows immediately from the combination of Propositions 5.1, 5.2 and 5.3.

\square

References

- [B] Bando, Shigetoshi, *Einstein Kähler metrics of negative Ricci curvature on open Kähler manifolds*. Kähler metric and moduli spaces, 105–136, Adv. Stud. Pure Math., **18-II**, Academic Press, Boston, MA, (1990).
- [CY1] Cheng, Shiu Yuen; Yau, Shing-Tung, *On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Fefferman’s equation*. Comm. Pure Appl. Math. **33** (1980), no. 4, 507–544.
- [CY2] Cheng, Shiu Yuen; Yau, Shing-Tung, *Inequality between Chern numbers of singular Kähler surfaces and characterization of orbit space of discrete group of $SU(2, 1)$* . Contemp. Math., **49**, Amer. Math. Soc., Providence, RI (1986).
- [F] Fefferman, Charles, *Monge-Ampère equations, the Bergman kernel, and geometry of pseudoconvex domains*. Ann. of Math. (2) **103** (1976), no. 2, 395–416.
- [GT] Gilbarg, David; Trudinger, Neil S. *Elliptic partial differential equations of second order*. Second edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **224**. Springer-Verlag, Berlin, (1983).
- [K] Kobayashi, Ryoichi, *Kähler-Einstein metric on an open algebraic manifold*. Osaka J. Math. **21** (1984), no. 2, 399–418.
- [LM] Lee, John M.; Melrose, Richard, *Boundary behaviour of the complex Monge-Ampère equation*. Acta Math. **148** (1982), 159–192.
- [S] Santoro, Bianca *Asymptotic Behavior of Complete Ricci-flat Kähler metrics on open manifolds*, Ph.D. thesis (2006).
- [TY1] Tian, Gang; Yau, Shing-Tung, *Complete Kähler manifolds with zero Ricci curvature. I*. J. Amer. Math. Soc. **3** (1990), no. 3, 579–609.
- [TY2] Tian, Gang; Yau, Shing-Tung, *Complete Kähler manifolds with zero Ricci curvature. II*. Invent. Math. **106** (1991), no. 1, 27–60.
- [TY3] Tian, Gang ; Yau, Shing-Tung, *Existence of Kähler-Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry*. Mathematical aspects of string theory, San Diego, Calif. (1986), 574–628
- [W] Wu, Damin, *Higher Canonical Asymptotics of Kähler-Einstein Metrics on Quasi-Projective Manifolds*, Ph.D. thesis (2005).

- [Y] Yau, Shing-Tung, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I.* Comm. Pure Appl. Math. **31** (1978), no. 3, 339–411

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